Insurance: Mathematics and Economics 64 (2015) 268–272

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Model points and Tail-VaR in life insurance

Michel Denuit [a], Julien Trufin [b,*]

[a] Institute of Statistics, Biostatistics and Actuarial Science, Université Catholique de Louvain (UCL), Louvain-la-Neuve, Belgium
[b] Department of Mathematics, Université Libre de Bruxelles (ULB), Bruxelles, Belgium

ARTICLE INFO

Article history:
Received December 2014
Received in revised form
May 2015
Accepted 5 June 2015
Available online 15 June 2015

Keywords:
Risk measures
Life insurance
Model points
Supermodular order
Convex order

ABSTRACT

Often, actuaries replace a group of heterogeneous life insurance contracts (different age at policy issue, contract duration, sum insured, etc.) with a representative one in order to speed the computations. The present paper aims to homogenize a group of policies by controlling the impact on Tail-VaR and related risk measures.

1. Introduction and motivation

Life insurance models are becoming more and more sophisticated under Solvency 2 regulation. European insurance companies are required to base their cash-flow projection on a policy-by-policy approach on the one hand, and to demonstrate the compliance of their internal model by carrying out additional testing on the other hand (see EIOPA, 2010). In particular, one of the validation tools recommended by the regulator is sensitivity testing, which consists in estimating the impact on the model outcomes of various changes in the underlying risk factors. Next to the baseline runs, insurers are then invited to conduct sensitivity analyses. Usually, all those studies need to be performed within tight deadlines. However, the use of Monte-Carlo simulations based on a policy-by-policy approach often leads to large running times (up to several days for the entire portfolio with the currently available computing power). Saving time when running the models thus appears to be an issue of major importance in life insurance.

A way to address this problem is to rely on grouping methods. Under certain conditions, the regulator permits the projection of future cash-flows based on suitable model points. We refer the reader to EIOPA (2010) for extensive details. The basic idea is to aggregate policies into homogeneous groups and to replace the group of contracts with a representative insurance policy in order to speed the simulation process. In this paper, we aim to homogenize a group of policies by controlling the impact on Tail-Value-at-Risk (Tail-VaR).

Related problems have already been considered in the actuarial literature. For instance, Frostig (2001) compared a heterogeneous portfolio composed of individual risks that are independent but not identically distributed with two homogeneous portfolios in which the risks are independent and identically distributed. The first homogeneous portfolio considered by Frostig (2001) is made of risks that are mixtures with equal weights of the risks in the heterogeneous portfolio and leads to an upper bound for the Tail-VaR of the heterogeneous portfolio. The second one consists of risks that are the average of the risks in the heterogeneous portfolio and turns out to be a lower bound.

Here, we use a simpler approach to obtain the upper bound derived in Frostig (2001) using a general comparison result obtained by Denuit and Müller (2002). Also, relying on this upper bound, we show how to build conservative model points with respect to Tail-VaR in a life insurance context. Finally, we improve the lower bound obtained in Frostig (2001) and we discuss various approximations.

The remainder of this paper is organized as follows. Section 2 recalls useful definitions and makes the problem under investigation more formal in terms of stochastic dominance rules and risk measures. Section 3 makes the connection with random sampling and mixture models. The result of Denuit and Müller (2002) is recalled and applied to derive stochastic inequalities among different sampling strategies. Section 4 applies these results to the derivation of model points. Several inequalities are derived in Sections 4

* Corresponding author.
E-mail address: julien.trufin@ulb.ac.be (J. Trufin).

http://dx.doi.org/10.1016/j.insmatheco.2015.06.002

and 5 to illustrate the impact of different aggregation procedures on actuarial indices in accordance with the stop-loss order (such as stop-loss premiums, Tail-VaRs, and spectral risk measures with concave distortions).

2. Stochastic dominance, risk measures and the problem of interest

2.1. Stochastic dominance rules

Before setting up the scene, we recall the definition of the stochastic dominance and of the convex order. We refer the interested reader, e.g., to Müller and Stoyan (2002), Denuit et al. (2005) or Shaked and Shanthikumar (2007) for more details.

Given two random variables X_1 and X_2 with respective distribution functions F_{X_1} and F_{X_2}, X_1 precedes X_2 in the usual stochastic order, denoted as $X_1 \preceq_{ST} X_2$, if

$$F_{X_1}(x) \geq F_{X_2}(x) \quad \text{for all } x,$$

or equivalently if

$$\overline{F}_{X_1}(x) \leq \overline{F}_{X_2}(x) \quad \text{for all } x,$$

where $\overline{F}_{X_1} = 1 - F_{X_1}$ and $\overline{F}_{X_2} = 1 - F_{X_2}$ are the excess, or survival functions corresponding to F_{X_1} and F_{X_2}, respectively. The latter is also equivalent to the inequality $\mathbb{E}[h(X_1)] \leq \mathbb{E}[h(X_2)]$ for any non-decreasing function h such that the expectations exist.

The usual stochastic order compares the sizes of the risks and translates in mathematical terms the concept of "being smaller than". The convex order focuses on the variabilities and enables the actuary to compare two risks with identical means. For two random variables X_1 and X_2 such that $\mathbb{E}[X_1] = \mathbb{E}[X_2]$, X_1 precedes X_2 in the convex order, denoted as $X_1 \preceq_{CX} X_2$, when

$$\int_x^{\infty} \overline{F}_{X_1}(u)\,du \leq \int_x^{\infty} \overline{F}_{X_2}(u)\,du \quad \text{for all } x. \tag{2.1}$$

The inequality in (2.1) can be equivalently written as

$$\mathbb{E}[(X_1 - x)_+] \leq \mathbb{E}[(X_2 - x)_+] \quad \text{for all } x. \tag{2.2}$$

From (2.2) it follows that $X_1 \preceq_{CX} X_2$ if and only if $\mathbb{E}[h(X_1)] \leq \mathbb{E}[h(X_2)]$ for all convex functions h, provided the expectations exist.

2.2. Corresponding risk measures

The stochastic order relations $\preceq_{ST}$ and $\preceq_{CX}$ can be defined by means of risk measures. Recall that the Value-at-Risk (VaR) of X_1 at probability level p is just the pth quantile of X_1, that is,

$$\text{VaR}[X_1; p] = F_{X_1}^{-1}(p) = \inf\{x \in \mathbb{R} | F_{X_1}(x) \geq p\}.$$

Then, it is easily deduced that

$$X_1 \preceq_{ST} X_2 \Leftrightarrow \text{VaR}[X_1; p] \leq \text{VaR}[X_2; p] \quad \text{for all probability levels } p.$$

So, $\preceq_{ST}$-inequalities can easily be interpreted as inequalities between VaRs, or more generally between weighted averages of VaRs (the so-called spectral risk measures, the weights being defined from distortion functions).

Besides VaRs, Tail-VaRs also play an important role in risk management, measuring the risk in the right tail. Specifically, the Tail-VaR of X_1 at probability level p is an average of the VaRs from that level on, i.e.

$$\text{TVaR}[X_1; p] = \frac{1}{1-p} \int_p^1 F_{X_1}^{-1}(\pi)\,d\pi.$$

Then, it can be shown that given two risks with equal means,

$$X_1 \preceq_{CX} X_2 \Leftrightarrow \text{TVaR}[X_1; p] \leq \text{TVaR}[X_2; p]$$

for all probability levels p.

Thus, $\preceq_{CX}$-inequalities can be interpreted as inequalities between TVaRs, or more generally between spectral risk measures with appropriate distortion functions.

2.3. Link to the problem of interest

Consider independent risks $X_1, \ldots, X_n$ causing an aggregate loss amount $X_1 + X_2 + \cdots + X_n$. Let F_i be the distribution function of X_i and let us assume that the X_is are ranked in increasing magnitude, i.e. the stochastic inequalities

$$X_1 \preceq_{ST} X_2 \preceq_{ST} \ldots \preceq_{ST} X_n$$

hold true. Our aim is to build two sets of independent and identically distributed random variables, henceforth denoted as $X_1^+, \ldots, X_n^+$ and $X_1^-, \ldots, X_n^-$, such that

$$\begin{aligned} X_1^- + X_2^- + \cdots + X_n^- &\preceq_{CX} X_1 + X_2 + \cdots + X_n \\ &\preceq_{CX} X_1^+ + X_2^+ + \cdots + X_n^+. \end{aligned} \tag{2.3}$$

3. Methodological results

The derivation of the upper bound in (2.3) turns out to be related to repeated sampling schemes from a given population. Mixtures of distributions are involved and this is why we consider in this section conditionally independent random variables, given a mixing random vector. Specifically, consider the random vector $(Z_1, \ldots, Z_n)$ with conditional distribution depending on a mixing vector parameter $(\Theta_1, \ldots, \Theta_n)$ as follows:

P1 Component Z_i depends only on Θ_i, i.e. the identity

$$\begin{aligned} \Pr[Z_i \leq t | \Theta_1 = \theta_1, \ldots, \Theta_n = \theta_n] &= \Pr[Z_i \leq t | \Theta_i = \theta_i] \\ &= F_i(t|\theta_i) \end{aligned}$$

holds for every $i \in \{1, \ldots, n\}$, where $F_i(\cdot|\theta_i)$ is the conditional distribution function of the ith component, given $\Theta_i = \theta_i$.

P2 The components $Z_1, \ldots, Z_n$ are conditionally independent, i.e. the identity

$$\begin{aligned} &\Pr[Z_1 \leq t_1, \ldots, Z_n \leq t_n] \\ &= \int_{\theta_1} \ldots \int_{\theta_n} \prod_{i=1}^n F_i(t_i|\theta_i) d\theta_1 \ldots d\theta_n \end{aligned}$$

holds for every $t_1, \ldots, t_n$.

Unconditionally, however, there may be dependence among the random variables $Z_1, \ldots, Z_n$ induced by the dependence structure of $(\Theta_1, \ldots, \Theta_n)$. Denuit and Müller (2002) investigated how the distribution of $(\Theta_1, \ldots, \Theta_n)$ affects the distribution of $(Z_1, \ldots, Z_n)$, especially how the dependence structure of $(Z_1, \ldots, Z_n)$ depends on the one of $(\Theta_1, \ldots, \Theta_n)$. We recall their result in the next property. To this end, we need the supermodular order. Let $\boldsymbol{S} = (S_1, \ldots, S_m)$ and $\boldsymbol{T} = (T_1, \ldots, T_m)$ be two random vectors where, for each i, S_i and T_i have the same marginal distributions. Then, $\boldsymbol{S}$ is less than $\boldsymbol{T}$ under supermodular order, denoted $\boldsymbol{S} \preceq_{SM} \boldsymbol{T}$, if $\mathbb{E}[\phi(\boldsymbol{S})] \leq \mathbb{E}[\phi(\boldsymbol{T})]$ for all supermodular functions ϕ, given that the expectations exist. Recall that a function $\phi : \mathbb{R}^m \to \mathbb{R}$ is supermodular if

$$\begin{aligned} &\phi(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_m) \\ &\quad - \phi(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_m) \\ &\geq \phi(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_m) \\ &\quad - \phi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m) \end{aligned}$$

holds for all $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $1 \leq i \leq j \leq m$ and all $\varepsilon, \delta > 0$. See Marshall and Olkin (1979) for examples of supermodular functions. The supermodular order is used to compare random vectors $\boldsymbol{S}$ and $\boldsymbol{T}$ with different levels of dependence.