Insurance: Mathematics and Economics 62 (2015) 28–41

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Personal finance and life insurance under separation of risk aversion and elasticity of substitution

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ARTICLE INFO

Article history:
Received June 2014
Received in revised form
January 2015
Accepted 23 February 2015
Available online 5 March 2015

Keywords:
Recursive utility
Lifetime uncertainty
Stochastic control
Generalized Hamilton–Jacobi–Bellman equation
Time-inconsistency
Certainty equivalents

ABSTRACT

In a classical Black–Scholes market, we establish a connection between two seemingly different approaches to continuous-time utility optimization. We study the optimal consumption, investment, and life insurance decision of an investor with power utility and an uncertain lifetime. To separate risk aversion from elasticity of inter-temporal substitution, we introduce certainty equivalents. We propose a time-inconsistent global optimization problem, and we present a verification theorem for an equilibrium control. In the special case without mortality risk, we discover that our optimization approach is equivalent to recursive utility optimization with Epstein–Zin preferences in the sense that the two approaches lead to the same result. We find this interesting since our optimization problem has an intuitive interpretation as a global maximization of certainty equivalents and since recursive utility, in contrast to our approach, gives rise to severe differentiability problems. Also, our optimization approach can there be seen as a generalization of recursive utility optimization with Epstein–Zin preferences to include mortality risk and life insurance.

1. Introduction

In a classical Black–Scholes market, we establish a connection between two seemingly different approaches to continuous-time utility optimization for a certain-lived investor. One approach is recursive utility optimization with Epstein–Zin preferences, studied in Duffie and Epstein (1992) and Kraft and Seifried (2010) for general preferences. The other approach is non-linear expected power utility optimization with dynamic updating, studied in this paper for an uncertain-lived investor. This approach is apt for a set-up with mortality risk and utility from inheritance, and because of the established connection for a certain-lived investor, our approach can be seen as a generalization of the recursive utility approach to a set-up with mortality risk and life insurance.

Over time, the optimal consumption and investment decisions of a certain-lived investor have been treated in various papers. An important, early example is Merton (1971) who considers time-additive utility optimization in continuous time. Using dynamic programming techniques, the value function of the time-additive optimization problem can be characterized by a partial differential equation. The equation is called a Hamilton–Jacobi–Bellman equation, and it includes a term $u(c)$ where u is the investor's utility function for consumption and c is the consumption rate.

Richard (1975) generalized the work by Merton (1971) to include mortality risk and life insurance. The value function, V, of the generalized optimization problem is characterized by a partial differential equation similar to the original Hamilton–Jacobi–Bellman equation. The main alteration consists in addition of the term

$$\mu(t)\,\tilde{u}(b+x) - \mu(t)\,V(t,x), \tag{1}$$

where μ is the investor's mortality intensity, $\tilde{u}$ is the investor's utility function for inheritance, b is a term insurance sum paid out upon death, and x is wealth. Also, there is an effect on the wealth dynamics due to financing of the term insurance. We note that $\mu(t)\,\tilde{u}(b+x)$ can be interpreted as the investor's probability weighted utility gain associated with death. Similarly, $\mu(t)\,V(t,x)$ can be interpreted as the investor's probability weighted utility loss associated with death. The term in (1) is therefore the investor's probability weighted net-gain associated with death.

Unfortunately, time-additive utility has the disadvantage that it mixes preferences for risk and preferences for inter-temporal substitution. The recursive utility approach and our approach both deal with this problem, in two seemingly different ways.

Recursive utility is founded in discrete time, and it allows for separation of preferences for risk and preferences for inter-temporal substitution through a recursive definition, a (utility)

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http://dx.doi.org/10.1016/j.insmatheco.2015.02.006
0167-6687/

certainty equivalent and a time-aggregator. In Duffie and Epstein (1992), recursive utility is extended to continuous time where it is called stochastic differential utility. The link to discrete-time recursive utility is vague though, and in Kraft and Seifried (2010), the extension is refined and called continuous-time recursive utility. In both papers, the optimal consumption and investment decisions of a certain-lived investor are studied. The value function, V, of the recursive optimization problem is characterized by a Hamilton–Jacobi–Bellman equation (in the following 'pseudo-Bellman equation') where the term $u(c)$ is replaced by a term $f(c, V(t,x))$. Here, f is the normalized aggregator representing the investor's preferences. In particular, Epstein–Zin preferences are represented by the aggregator

$$f(c,V) = \theta\delta V\left(\left(\frac{c}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{\frac{1-\gamma}{\theta}} - 1\right).$$

The recursive optimization problem is less intuitive than the time-additive optimization problem, and to our knowledge, the literature contains no attempt to extend the recursive utility problem to a set-up with mortality risk and life insurance. However, inspired by the mortality extension in Richard (1975), it is natural to suggest a pseudo-Bellman equation where we combine $f(c,V)$ defined above with the additional term $\mu(t)\tilde{u}(b+x) - \mu(t)V(t,x)$.

For Epstein–Zin preferences, we present another suggestion—namely an alteration of the normalized aggregator (and no additional term). The altered aggregator arises from the following optimization approach: we consider an uncertain-lived investor with power utility. To separate preferences for risk and preferences for inter-temporal substitution, we introduce consumption certainty equivalents, and we propose a time-global optimization problem that is about maximizing an infinite sum of infinitesimally small certainty equivalents for future consumption and inheritance. The problem is non-linear in expectation, and consequently it is time-inconsistent in the sense that its solution does not obey Bellman's optimality principle. In other words: if we solve the problem at time 0 and apply the corresponding control up to a future time point $t > 0$, then at this future time point, the control is no longer optimal. For more on time-inconsistency, see e.g. Björk et al. (2014) or Björk and Murgoci (2010). To deal with the time-inconsistency, we search for an equilibrium control instead of a classical optimal control, and we present a verification theorem for a particular equilibrium control. The corresponding value function is characterized by a pseudo-Bellman equation where the term $f(c, V(t,x))$ is replaced by the term $\tilde{f}(t, c, x+b, V(t,x))$. Here, the altered aggregator $\tilde{f}$ is given by

$$\tilde{f}(t,c,y,V) = \theta\delta V\left(\left(\frac{c^{1-\gamma}}{V(1-\gamma)}\right)^{\frac{1}{\kappa}} + \left(\frac{\epsilon(t)\mu(t)y^{1-\gamma}}{V(1-\gamma)}\right)^{\frac{1}{\kappa}}\right)^{\frac{\kappa}{\theta}} - (\mu(t)+\theta\delta)V.$$

For a certain-lived investor (i.e. $\mu = 0$), the two aggregators f and $\tilde{f}$ coincide, and so our approach leads to the same result as recursive utility optimization with Epstein–Zin preferences, for a certain-lived investor. Because of this equivalence, the aggregator $\tilde{f}$ can be seen as a mortality extension of the normalized Epstein–Zin aggregator—that is, our approach can be seen as a generalization of the recursive utility approach with Epstein–Zin preferences to a set-up with mortality risk and life insurance. This proposal is supported by the fact that our optimization problem has an intuitive interpretation as a global maximization of certainty equivalents, both with and without mortality risk. Furthermore, our approach is a generalization of the time-additive utility optimization in Richard (1975) to time-non-additive power utility.

Recursive utility is considered as a standard way to separate risk aversion from elasticity of inter-temporal substitution. We provide a new way to formalize such a separation where, first, risk aversion forms certainty equivalents and, then, elasticity of substitution forms time-global preferences. Yet, a completely different approach to the separation is suggested in Kihlstrom (2009). In discrete time, he suggests to formalize a separation where, first, elasticity of substitution forms time-global preferences and, then, risk aversion forms one certainty equivalent. Since his formalization is not immediately tractable with our method, future research should address further the relation between Kihlstrom's approach, our approach, and recursive utility.

We emphasize that our optimization problem is not a special case of Björk and Murgoci (2010) as our objective function has a considerably different form. In particular, their result about coincidence of solutions for certain time-consistent and time-inconsistent problems does not explain the equivalence between our approach and recursive utility optimization with Epstein–Zin preferences. Also, we wish to focus on our specific investor problem and not on time-consistency in general, so we do not go into details on the game-theoretic equilibrium approach.

We work in a simple Black–Scholes market because we wish to study the qualitative structures of the solution to our optimization problem. We then avoid drowning our key insights in notation and multidimensionality, and we avoid resorting to numerical optimization. For qualitative insight, sticking to a simple model remains efficient.

Structure of the paper

In Section 2, we propose an optimization problem and introduce the concept of equilibrium controls. We present a verification theorem for a particular equilibrium control, and we derive closed-form expressions for the control and the corresponding value function. Finally, we compare our results to Richard (1975).

In Section 3, we give a short introduction to recursive utility, and we demonstrate the similarity of our pseudo-Bellman equation and the pseudo-Bellman equation in Duffie and Epstein (1992). Also, we outline perspectives of the established equivalence.

In Section 4, we derive a stochastic differential equation for the optimal consumption rate from Section 2, and we construct numerical examples to illustrate how it differs from the optimal consumption rate from time-additive utility. The numerical examples all arise from the special case without market risk.

2. Optimization problem

2.1. Set-up

We consider an investor making decisions concerning consumption, investment, and life insurance in continuous time. We adopt the classical survival model, and by N and $I = 1 - N$, we indicate whether the investor is dead or alive at a given point in time (e.g. $N(t) = 1$ if the investor is dead at time t). We treat N and I as stochastic processes on an abstract probability space $(\Omega, \mathcal{F}, P)$, and we model the death of the investor by a mortality intensity μ, i.e.

$$P(I(t)=1) = P(I(s)=1 : s\in[0,t]) = e^{-\int_0^t \mu(v)\,\mathrm{d}v}, \quad t \ge 0.$$

The investor has access to a classical Black–Scholes market consisting of a bank account, B, with risk free short rate r, and a stock, S, with excess return λ and volatility σ. The asset prices are described by the stochastic differential equations (SDEs)

$$\mathrm{d}B(t) = B(t)\,r\,\mathrm{d}t, \quad t\ge 0, \qquad B(0)=1,$$
$$\mathrm{d}S(t) = S(t)\left[(r+\lambda)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\right], \quad t \ge 0, \qquad S(0)=s_0,$$