J. Math. Anal. Appl. 418 (2014) 612–625

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A new characterization of comonotonicity and its application in behavioral finance

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ARTICLE INFO

Article history:
Received 20 August 2013
Available online 25 March 2014
Submitted by Goong Chen

Keywords:
Comonotonicity
Behavioral finance
Quantile formulation
Atomless/non-atomic
Pricing kernel
Cumulative prospect theory
Rank-dependent utility theory
Economic equilibrium model

ABSTRACT

It is well-known that an $\mathbb{R}^n$-valued random vector $(X_1, X_2, \cdots, X_n)$ is comonotonic if and only if $(X_1, X_2, \cdots, X_n)$ and $(Q_1(U), Q_2(U), \cdots, Q_n(U))$ coincide *in distribution*, for *any* random variable U uniformly distributed on the unit interval $(0,1)$, where $Q_k(\cdot)$ are the quantile functions of X_k, $k = 1, 2, \cdots, n$. It is natural to ask whether $(X_1, X_2, \cdots, X_n)$ and $(Q_1(U), Q_2(U), \cdots, Q_n(U))$ can coincide *almost surely* for *some* special U. In this paper, we give a positive answer to this question by construction. We then apply this result to a general behavioral investment model with a law-invariant preference measure and develop a universal framework to link the problem to its quantile formulation. We show that any optimal investment output should be anti-comonotonic with the market pricing kernel. Unlike previous studies, our approach avoids making the assumption that the pricing kernel is atomless, and consequently, we overcome one of the major difficulties encountered when one considers behavioral economic equilibrium models in which the pricing kernel is a yet-to-be-determined unknown random variable. The method is applicable to general models such as risk sharing model.

1. Introduction

The concept of comonotonicity has wide applications in actuarial science and financial risk management, see, e.g., Dhaene, Denuit, Goovaerts, Kaas, and Vyncke [5,6], Di Nunno and Øksendal [7], Deelstra, Dhaene, and Vanmaele [4]. It mainly refers to the perfect positive dependence between the components of a random vector, essentially saying that they can be represented as increasing functions of a single random variable. In this paper, we will show that these functions can be specified as their individual's quantile functions and this random variable specified as a random variable uniformly distributed on the unit interval $(0,1)$ that is comonotonic with the components of the random vector.

This paper consists of two parts. The first part is dedicated to studying a new characterization of comonotonic random vectors. In the second part, we apply this characterization to a general behavioral investment problem with a law-invariant preference measure and develop a universal framework to link

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http://dx.doi.org/10.1016/j.jmaa.2014.03.053
0022-247X/

the problem to its quantile formulation, which overcomes one of the major difficulties encountered when one considers a behavioral economic equilibrium model with a law-invariant preference measure where the classical dynamic programming and probabilistic approaches do not work.

We start by introducing the concept of comonotonicity for a random vector. Let us first recall the definition and the characterizations of a comonotonic random vector. For the rest part of this paper, all the random variables/vectors are in the same given probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 1. An $\mathbb{R}^n$-valued random vector $(X_1, X_2, \cdots, X_n)$ is comonotonic if there exits a set $\widehat{\Omega}\times\widehat{\Omega} \subseteq \Omega\times\Omega$, with full measure, such that

$$\big(X_i(\omega') - X_i(\omega)\big)\big(X_j(\omega') - X_j(\omega)\big) \geqslant 0, \quad \text{for all } \big(\omega', \omega\big) \in \widehat{\Omega} \times \widehat{\Omega},\ i, j \in \{1, 2, \cdots, n\}.$$

We denote by $\mathbb{U}$ the set of all random variables uniformly distributed on the unit interval $(0, 1)$ in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. In order to get a new characterization of comonotonic random vector, it is necessary to assume, throughout this paper, that

Assumption 1. The set $\mathbb{U}$ is not empty.

Define the quantile function $Q_X(\cdot)$ of an $\mathbb{R}$-valued random variable X as the right-continuous inverse function of its cumulative distribution functions (cdf) $F_X(\cdot)$, that is,

$$Q_X(x) = \sup\{t \in \mathbb{R}:\ F_X(t) \leqslant x\}, \quad x \in (0, 1),$$

with convention $\sup\emptyset = -\infty$.

The following well-known result characterizes the comonotonic random vector (see, e.g., Dhaene, Denuit, Goovaerts, Kaas, and Vyncke [5,6]).

Theorem 1. *An $\mathbb{R}^n$-valued random vector $\mathcal{X} = (X_1, X_2, \cdots, X_n)$ is comonotonic if and only if one of the following conditions holds:*

1. *For any vector $\underline{\mathbf{x}} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, we have*

$$F_{\mathcal{X}}(\underline{\mathbf{x}}) = \min\big\{F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_n}(x_n)\big\},$$

   where $F_{\mathcal{X}}(\cdot)$ and $F_{X_k}(\cdot)$, $k = 1, 2, \cdots, n$, are the cumulative distribution functions (cdf) of $\mathcal{X}$ and X_k, $k = 1, 2, \cdots, n$, respectively;
2. *For any $U \in \mathbb{U}$, we have*

$$\mathcal{X} \overset{\mathrm{d}}{=} \big(Q_1(U), Q_2(U), \cdots, Q_n(U)\big),\text{[1]}$$

   where $Q_k(\cdot)$ are the quantile functions of X_k, $k = 1, 2, \cdots, n$;
3. *There exist a random variable Y and non-decreasing functions $f_k(\cdot)$, $k = 1, 2, \cdots, n$, such that*

$$\mathcal{X} \overset{\mathrm{d}}{=} \big(f_1(Y), f_2(Y), \cdots, f_n(Y)\big);$$

4. *Let $Y = X_1 + X_2 + \cdots + X_n$. There exist non-decreasing functions $f_k(\cdot)$, $k = 1, 2, \cdots, n$, such that*

$$\mathcal{X} = \big(f_1(Y), f_2(Y), \cdots, f_n(Y)\big).\text{[2]}$$

[1] We write $X \overset{\mathrm{d}}{=} Y$ if random variables/vectors X and Y are identically distributed.
[2] For any two random variables/vectors X and Y, we write $X = Y$ if $\mathbf{P}(X = Y) = 1$.