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Growth optimal investment in discrete-time markets with proportional transaction costs

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ABSTRACT

We investigate how and when to diversify capital over assets, i.e., the portfolio selection problem, from a signal processing perspective. To this end, we first construct portfolios that achieve the optimal expected growth in i.i.d. discrete-time two-asset markets under proportional transaction costs. We then extend our analysis to cover markets having more than two stocks. The market is modeled by a sequence of price relative vectors with arbitrary discrete distributions, which can also be used to approximate a wide class of continuous distributions. To achieve the optimal growth, we use threshold portfolios, where we introduce a recursive update to calculate the expected wealth. We then demonstrate that under the threshold rebalancing framework, the achievable set of portfolios elegantly form an irreducible Markov chain under mild technical conditions. We evaluate the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized yielding the growth optimal portfolio under proportional transaction costs in i.i.d. discrete-time two-asset markets. As a widely known financial problem, we also solve the optimal portfolio selection problem in discrete-time markets constructed by sampling continuous-time Brownian markets. For the case that the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator that is also incorporated in the optimization framework in our simulations.

1. Introduction

The problem of how and when an investor should diversify capital over various assets, whose future returns are yet to be realized, is extensively studied in various different fields from signal processing [1,2,12,28–30,33,35] and financial engineering [25,26] to machine learning [13,34] and information theory [9]. Naturally, this is one of the most important financial applications due to the amount of money involved. However, the recent financial crisis demonstrated that there is a significant room for improvement in this field by sound signal processing methods [12,30], which is the main goal of this paper. In this paper, we investigate how and when to diversify capital over assets, i.e., the portfolio selection problem, from a signal processing perspective and provide portfolio selection strategies that maximize the expected cumulative wealth in discrete-time markets under proportional transaction costs.

In particular, we study an investment problem in markets that allows trading at discrete periods, where the discrete period is arbitrary, e.g., it can be seconds, minutes or days [24]. Furthermore, the market levies transaction fees for both selling and buying an asset proportional to the volume of trading at each transaction, which accurately models a broad range of financial markets [3,24]. In our discussions, we first consider markets with two assets. Two-stock markets are extensively studied in financial literature and are shown to accurately model a wide range of financial applications [24] such as the well-known "Stock and Bond Market" [24]. We then extend our analysis to markets having more than two assets, i.e., m-stock markets, where m is arbitrary.

Following the extensive literature [9,19,24–26,33], the market is modeled by a sequence of price relative vectors, say $\{\boldsymbol{X}(n)\}_{n\geq 1}$, $\boldsymbol{X}(n) \in [0,\infty)^m$, where each entry of $\boldsymbol{X}(n)$, i.e., $X_i(n) \in [0,\infty)$, is the ratio of the closing price to the opening price of the ith stock per investment period. Hence, each entry of $\boldsymbol{X}(n)$ quantifies the gain (or the loss) of that asset at each investment period. The sequence of price relative vectors is assumed to have an i.i.d. "dis-

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crete" distribution [24–26,33], however, the discrete distributions on the vector of price relatives are arbitrary. In this sense, the corresponding discrete distributions can approximate a wide class of continuous distributions on the price relatives that satisfy certain regularity conditions by appropriately increasing the size of the discrete sample space. We first assume that the discrete distributions on the price relative vectors are known and then extend our analysis to cover the case, where the underlying distributions are unknown. We emphasize that the i.i.d. assumption on the sequence of price relative vectors is shown to hold in most realistic markets [14,24].

At each investment period, the diversification of the capital over the assets is represented by a portfolio vector $\boldsymbol{b}(n)$, where $\forall i \in \{1,\ldots,m\}$, $b_i(n)$ represents the ratio of the capital invested in the ith asset at investment period n, i.e., we have $\sum_{i=1}^{m} b_i(n) = 1$, where $0 \leq b_i(n) \leq 1$. As an example, if we invest using $\boldsymbol{b}(n)$, we earn (or lose) $\boldsymbol{b}^T(n)\boldsymbol{X}(n)$ at the nth investment period, after $\boldsymbol{X}(n)$ is revealed. Given that we start with one dollar, after an investment period of N days, we have the wealth growth $\prod_{n=1}^{N} \boldsymbol{b}^T(n)\boldsymbol{X}(n)$. Under this general market model, we provide algorithms that maximize the expected growth over any period N by using "threshold rebalanced portfolios" (TRPs), which are shown to yield optimal growth in general i.i.d. discrete-time markets [14].

In [9], Cover et al. showed that the portfolio that achieves the maximal growth is a constant rebalanced portfolio (CRP) in i.i.d. discrete-time markets, under certain assumptions on the sequence of price relatives and without any transaction costs. A CRP is a portfolio investment strategy, where the fraction of wealth invested in each stock is kept constant at each investment period. A problem extensively studied in this framework is to find sequential portfolios that asymptotically achieve the wealth of the best CRP tuned to the underlying sequence of price relatives. Several sequential algorithms are introduced to achieve the performance of the best CRP (such as [9,13,16,34]) with different convergence rates and different performances on historical data sets. In [3], sequential algorithms that achieve the performance of the best CRP under transaction costs are introduced. However, we emphasize that keeping a CRP may require extensive trading due to a possible rebalancing at each investment period which deems CRPs (even the best CRP) ineffective in realistic markets even under mild transaction costs [19].

In continuous-time markets, however, it has been shown that under transaction costs, the optimal portfolios that achieve the maximal wealth are certain class of "no-trade zone" portfolios [7,11,32]. In simple terms, a no-trade zone portfolio has a compact closed set and a rebalancing occurs if the current portfolio breaches this set, otherwise no rebalancing occurs. Clearly, such a no-trade zone portfolio may avoid hefty transaction costs since it can limit excessive rebalancing by defining appropriate no-trade zones. Analogous to continuous time markets, it has been shown in [14] that in two-asset i.i.d. markets under proportional transaction costs, compact no-trade zone portfolios are optimal such that they achieve the maximal growth under certain assumptions on the sequence of price relatives. In two-asset markets, the compact no-trade zone is represented by thresholds, e.g., if at investment period n, the portfolio is given by $\boldsymbol{b}(n) = [b(n), \ (1-b(n))]^T$, where $0 \leq b(n) \leq 1$, then rebalancing occurs if $b(n) \notin (\alpha, \beta)$, given the thresholds α, β, where $0 \leq \alpha \leq \beta \leq 1$. Similarly, the interval (α, β) can be represented using a target portfolio b and a region around it, i.e., $(b-\epsilon, b+\epsilon)$, where $0 \leq \epsilon \leq \min\{b, 1-b\}$ such that $\alpha = b - \epsilon$ and $\beta = b + \epsilon$. Extension of TRPs to markets having more than two stocks is straightforward and explained in Section 3.2.

However, how to construct the no-trade zone portfolio, i.e., how to select the thresholds that achieve the maximal growth, has not yet been solved except in elementary scenarios [14]. In [15], a universal algorithm that asymptotically achieves the performance of the best TRP tuned to the underlying sequence of price relatives is introduced. This algorithm leverages Bayesian type weighting from [9] inspired from universal source coding and requires no statistical assumptions on the sequence of price relatives. In similar lines, various different universal algorithms are introduced that achieve the performance of the best expert in different competition classes in [1,2,17–20]. Although the performance guarantees in [1,2,15,18,19]) are derived without any stochastic assumptions, these methods are highly conservative due to the worst case scenario optimization, i.e., they are only optimal in an asymptotical sense. However, the order of such performance upper bounds may not be negligible in actual financial markets [6,20], even though they may be neglected in source coding applications (where these algorithms are inspired from). We demonstrate that our algorithm readily outperforms these universal methods over historical data.

Our main contributions are as follows. We first consider two-asset markets and recursively evaluate the expected achieved wealth of a threshold portfolio for any b and ϵ over any investment period. We then extend this analysis to markets having more than two-stocks. We next demonstrate that under the threshold rebalancing framework, the achievable set of portfolios form an irreducible Markov chain under mild technical conditions. We evaluate the corresponding stationary distribution of this Markov chain, which provides a natural and efficient method to calculate the cumulative expected wealth. Subsequently, the corresponding parameters are optimized using a brute force approach yielding *the growth optimal investment portfolio under proportional transaction costs in i.i.d. discrete-time* two-asset markets. We note that for the case with the irreducible Markov chain, which covers practically all scenarios in the realistic markets, the optimization of the parameters is offline and carried out only once. However, for the case with recursive calculations, the algorithm has an exponential computational complexity in terms of the number of states. However, in our simulations, we observe that a reduced complexity form of the recursive algorithm that keeps only a constant number of states by appropriately pruning certain states provides nearly identical results with the "optimal" algorithm. Furthermore, as a well studied problem, we also solve optimal portfolio selection in discrete-time markets constructed by sampling continuous-time Brownian markets [24]. When the underlying discrete distributions of the price relative vectors are unknown, we provide a maximum likelihood estimator to estimate the corresponding distributions that is incorporated in the optimization framework in the Simulations section. For all these approaches, we also provide the corresponding complexity bounds.

The organization of the paper is as follows. In Section 2, we briefly describe our discrete-time stock market model with discrete price relatives and symmetric proportional transaction costs. In Section 3, we start to investigate TRPs, where we first introduce a recursive update in Section 3.1 for a market having two-stocks. Generalization of the iterative algorithm to the m-asset market case is provided in Section 3.2. We then show that the TRP framework can be analyzed using finite state Markov chains in Section 3.4 and Section 3.5. The special Brownian market is analyzed in Section 3.6. The maximum likelihood estimator is derived in Section 4. We simulate the performance of our algorithms in Section 5 and conclude the paper with certain remarks in Section 6.

2. Problem description

We consider discrete-time stock markets under transaction costs. We first consider a market with two stocks and then extend the analysis to markets having more than two stocks. We model the market using a sequence of price relative vectors $\mathbf{X}(n)$.