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Optimal investment policy in the time consistent mean–variance formulation[☆]

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ABSTRACT

As a necessary requirement for multi-period risk measure, time consistency can be examined from two aspects: dynamic risk measure and optimal investment policy. In this paper, we first study the relationship between the time consistency of dynamic risk measure and the time consistency of optimal investment policy and obtain the following conclusions: if the dynamic risk mapping is time consistent and monotone, then the corresponding optimal investment policy satisfies the time consistency requirements; however, if the dynamic risk mapping is time consistent but not monotone, then the time consistency requirements of an optimal investment policy will no longer be satisfied. Since the variance operator does not satisfy the smoothing property, the optimal investment policy derived from the existing multi-period mean–variance model is not time consistent. To overcome this shortcoming, we propose the notation of a separable expected conditional mapping and then construct a time consistent dynamic mean–variance model. We prove that the optimal investment policy derived from our model is time consistent. Moreover, for two cases with or without a riskless asset, we obtain the time consistent analytical optimal investment policy and the mean–variance efficient frontier of the new model with the self-financing constraint. Finally, numerical results illustrate the flexibility and superiority of our multi-period mean–variance model and the optimal investment policy over those in the literature.

1. Introduction

It is now accepted that the time consistency should be a necessary requirement for the multi-period risk measure and relevant portfolio selection problems. In general, we can discuss the time consistency from two aspects: dynamic risk measure and optimal investment policy.

The time consistency of dynamic risk measure aims at characterizing the relationship among risks at individual stages. Research about time consistency can be traced back to Koopmans (1960), Kreps and Porteus (1978), and Epstein and Zin (1989); these studies are mainly about the time consistency of preferences in terms of the utility function. Wang (1999) first proposed the notion of time consistency of risk measure, whose main idea can be simply described as follows: for two investment positions X and Y, if X is riskier than Y under a specific risk measure at any time in the future, then X is riskier than Y under the same measure at present. Inspired by this idea, Roorda et al. (2005) and Artzner et al. (2007) consider the multi-period coherent risk measures and propose a dynamic time consistency similar to that of Wang (1999). As an extension to that, Roorda et al. (2005) and Roorda and Schumacher (2007) further propose two weaker time consistencies, sequential time consistency and conditional time consistency. For the dynamic convex risk measure, similar definitions as that of the dynamic time consistency are also introduced, see, for example, Detlefsen and Scandolo (2005), Föllmer and Penner (2006) and the references therein. Moreover, Cheridito and Kupper (2009) consider the dynamic utility function and propose a time consistency which is similar to dynamic time consistency. In all the above papers, the authors only pay attention to the terminal wealth. However, for the multi-period investment problem, investors are usually more concerned with investment positions at intermediate periods. Riedel (2004) thus considers the cash flow over the entire investment horizon and proposes the notion of time consistency (also called dynamic time consistency) which can be expressed as follows: for any two cash flows A and B, if they have the same value under a given risk measure at period $t+1$ in the future and A is the same as B at the stage between t and $t+1$, then A and B have the same value under the measure at the present period t. Similar notions are also proposed in Cheridito et al. (2006) and Ruszczyński (2010). If the dynamic risk measure is dynamic time consistent, then, it can be expressed recursively by the corresponding single-period risk measure. This conclusion can be found in Detlefsen and Scandolo (2005) for dynamic convex risk measures and Roorda and Schumacher (2007) for dynamic coherent

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risk measures, respectively. For this reason, in some papers, recursive relations are adopted to define the time consistency of dynamic risk measures, see, for example, Cheridito et al. (2006), and Jobert and Rogers (2008).

Compared with the time consistency of dynamic risk measures, research about the time consistency of an optimal investment policy are few. Time consistency of the optimal investment policy does not hold even for some popular risk measures; for instance, Boda and Filar (2006) has pointed out that the optimal investment policy under VaR or CVaR are not time consistent. Inspired by the optimality principle of dynamic programming, Boda and Filar (2006) propose the notion of time consistency about the optimal investment policy through introducing the following two requirements:

(A1) For the optimal investment problem with respect to some risk measure, the corresponding policy constituted by the stage-wise optimal decisions recursively obtained by the dynamic programming method is also the optimal investment policy to the whole problem. In short, local optimum is also globally optimum.

(A2) For the optimal investment policy of the whole optimal investment problem, the sub-policy is also the optimal policy for the corresponding sub-problem, which is actually Bellman's optimality principle.

At present, most studies about the time consistency of an optimal investment policy are mainly referred to (A2) (see, for example, Cui et al., 2012, Wang and Forsyth, 2011), which is also called time consistent. However, in order to ensure the reasonability of solving the relevant optimal portfolio problem by the dynamic programming method, (A1) is an essential requirement. Therefore, in this paper, we will adopt both (A1) and (A2) as the definition of time consistency of the optimal investment policy.

Shapiro (2009) proposes another understanding about time consistency of the optimal policy, the main idea is that the optimal strategy in any state should not depend on the scenarios which cannot happen in the future. For some optimization problems, even if we can derive the corresponding dynamic programming equations, the above time consistency might not be satisfied. This happens, for example, when some decision variables depend on the whole scenario tree. Therefore, the above notion about the time consistency of the optimal investment policy is similar to Bellman's optimality principle but not the same.

In most cases, even the optimal investment strategy derived from relatively simple portfolio selection problems are not necessarily time consistent in the sense of Shapiro (2009). One can refer to Boda and Filar (2006) for the time consistency issue of the optimal investment policy of portfolio selection problems under VaR or CVaR.

As two aspects of time consistency, there should exist some connection between the time consistency of dynamic risk measure and the time consistency of optimal investment policy. This relationship has not been clearly examined in the literature. Due to this, we firstly show in this paper: if the dynamic risk measure satisfies the monotonicity and dynamic time consistency in the sense of Wang (1999), then the corresponding optimal investment policy satisfies the time consistency requirements (A1) and (A2) proposed by Boda and Filar (2006). However, if the dynamic risk measure is time consistent but not monotone, then the derived optimal investment policy only satisfies Bellman's optimality principle, that is, (A2).

It is well known that, in order to find the optimal portfolio, one has to face a dilemma: to reduce risk or to increase the investment return. Markowitz (1952) proposes the first systematic method to deal with this dilemma, and his seminal work is considered to be the foundation of modern finance theory. Following Markowitz's mean–variance (MV) model, a great number of researches about the optimal portfolio selection have been aroused, see, for example, the review paper by Steinbach (2001) and 208 references therein. Merton (1972) obtains the analytical optimal solution of the static MV model with no short-selling constraints. The single-period MV model has been widely studied and applied. However, it is powerless when the investor has a particular requirement at a specific time point in the future. Therefore, the static model is naturally extended to the multi-period case. The earliest research about the multi-period problem can be traced back to Tobin (1965). Merton (1969) considers the portfolio selection problem of the continuous-time MV problem. Dumas and Luciano (1991) further study the multi-period problem with transaction costs. Unfortunately, all these works do not provide an explicit solution or an efficient method to determine the optimal investment strategy. With only the self-financing constraint, Li and Ng (2000) solve the multi-period MV problem by embedding it into a separable parametric auxiliary problem and obtain the analytical optimal policy. In the same year, the continuous-time MV problem was also studied by Zhou and Li (2000). Furthermore, Li et al. (2001) consider the continuous-time MV model with no short-selling constraints. As an indispensable ingredient of risk control, Zhu et al. (2004) study the multi-period MV model with bankruptcy constraints and derive the analytical optimal investment policy. Bielecki et al. (2005) further consider the continuous-time MV model with bankruptcy control. In all the above papers about the multi-period MV problem and other relevant studies in Li et al. (1998), Yu et al. (2005), Yu et al. (2010), and Cui et al. (2012), it is assumed that the random returns of risky assets among different periods are statistically independent in order to derive an explicit expression for the optimal investment policy.

Since the variance operator does not satisfy the smoothing property as that of the expectation operator (Li and Ng, 2000), the analytical optimal investment policy derived from the above multi-period or continuous-time MV problems does not satisfy Bellman's optimality principle. To overcome this difficulty, Cui et al. (2012) propose a weak time consistency compared to Bellman's optimality principle, i.e., time consistency in efficiency. The main difference of this notion of time consistency from Bellman's optimality principle is that the sub-policy of an optimal policy is the optimal policy of the corresponding sub-problem where the trade-off parameter varies over time. On the other hand, imposing Bellman's optimality principle as a constraint in the continuous-time MV model, Wang and Forsyth (2011) compare the efficient frontiers obtained from the time consistent optimal policy and the pre-commitment optimal policy, respectively, when additional constraints, such as bankruptcy, no short-selling, are added to the problem.

Unfortunately, all the above studies and the current literature we have seen do not provide such a dynamic MV model that we can solve it analytically and obtain an optimal policy which satisfies the time consistency requirements (A1) and (A2). To this end, we propose in this paper a multi-period MV model with the self-financing constraint, the analytical optimal investment policy is then derived by using the dynamic programming technique and, more importantly, we show that the policy satisfies the time consistency requirements (A1) and (A2).

The rest of this paper is organized as follows. In the next section, we investigate the relationship between the time consistency of dynamic risk measure and the time consistency of the optimal investment policy. In Section 3, we propose a separable multi-period MV model and prove that the resulting optimal investment policy would satisfy the time consistency requirements (A1) and (A2). In Section 4, we firstly consider a security market with only risky assets and obtain the time consistent analytical optimal investment policy. The time consistent analytical optimal