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Time-consistent investment policies in Markovian markets: A case of mean–variance analysis ☆

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ABSTRACT

The optimal investment policy for a standard multi-period mean–variance model is not time-consistent because the variance operator is not separable in the sense of the dynamic programming principle. With a nested conditional expectation mapping, we develop an investment model with time consistency in Markovian markets. Furthermore, we examine the differences of the investment policies with a riskless asset from those without a riskless asset. Analytical solutions for time-consistent optimal investment policies and the resulting mean–variance efficient frontier are obtained. Finally, using numerical examples, we show that the optimal investment policy derived from our model is more efficient than that of the standard mean–variance model in which the trade-off is determined between the mean and variance of the terminal wealth.

1. Introduction

In a dynamic setting, investment risk changes with economic information flows that substantiate reasonable risk measurement over time. Characterizing the relationship of risk across time periods is a very important step in modeling dynamic investments. Financial economists call this relationship as time-consistency. In general, dynamic time consistency of a risk measure is defined so that, for two investment portfolios X and Y, if X is riskier than Y from a future time point, then X is riskier than Y from the current time point. Different versions of time consistency have been proposed in the literature. Early contributions, including Koopmans (1960), Kreps and Porteus (1978), and Epstein and Zin (1989), study the time consistency of preferences under the framework of utility maximization. Recently, more researchers, such as Roorda et al. (2005), Detlefsen and Scandolo (2005) and Föllmer and Penner (2006), concentrate on dynamic time consistency. Weber (2006) and Artzner et al. (2007) study the dynamic time consistency for a multi-period coherent risk measure.

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All discussions of dynamic time consistency in the aforementioned research work are subject to the limitation that risk is quantified by the terminal wealth of an investment. In reality, investors do care about not only the terminal wealth but also the cash-flows in all periods within the investment horizon. Up to date, dynamic time consistency in relation to cash-flows has been studied by Riedel (2004), Cheridito et al. (2006), and Ruszczyński (2010). There are two aspects in the study of dynamic time consistency, i.e., time consistency of a dynamic risk measure and the time consistency of a dynamic investment policy. According to Detlefsen and Scandolo (2005), if a risk measure satisfies the dynamic time consistency, then it can be recursively obtained through period by period backward induction. Similar results can be found in Roorda and Schumacher (2007), Ruszczyński (2010), Cheridito et al. (2006) and Jobert and Rogers (2008). Boda and Filar (2006) define the time consistency of an investment policy to simultaneously satisfy the following two conditions:

(TC1) The investment policy comprised of stage optimal decisions recursively obtained through the dynamic programming method is optimal for the whole horizon problem.
(TC2) A sub-policy of the optimal investment policy for a multi-period investment problem is also optimal for the associated sub-problem.

Since dynamic time-consistent risk measures can be determined recursively by single-period risk measures, research on dynamic time consistency of an investment policy has mainly focused on (TC2); see Cui et al. (2010) and Wang and Forsyth (2011). It is known that condition (TC1) holds if the single-period risk measure is monotonic. However, the monotonicity breaks down for the risk measure determined by the variance operator. So, even if we define recursively the dynamic variance measure by using single-period variances, we cannot solve the dynamic mean–variance (MV) investment problem by using the dynamic programming technique. Therefore, (TC1) and (TC2) are jointly referred to as time consistency in this paper.

Portfolio selection problems are concerned with the optimal allocation of an investor′s wealth to a basket of securities. Markowitz (1952) provides a basis for modern portfolio analysis with a single-period MV model. However, for the portfolio selection problem in practice, the financial position changes with new information over time. Therefore, it is meaningful to study multi-period or dynamic investment problems. The pioneering single-period MV model can be naturally extended to a dynamic setting; see Tobin (1965), Merton (1969), and Dumas and Luciano (1991). Nevertheless, it is difficult to explicitly determine the optimal investment policy in a multi-period setting due to the non-separability of the variance operator. Solutions to dynamic MV problems have not been rigorous until Li and Ng (2000). Since then, a number of papers have studied dynamic MV models. With only a self-financing constraint, Li and Ng (2000) embed the multi-period MV problem into a nested parametric auxiliary problem and obtain an analytical optimal policy and the associated MV efficient frontier. With a short-selling constraint, Li et al. (2001) study a continuous-time MV model. With bankruptcy constraints, Zhu et al. (2004) and Bielecki et al. (2005) study discrete-time and continuous-time MV models, respectively.

In the aforementioned literature for multi-period MV models, it is assumed that asset returns in different time periods are independent. This assumption is problematic in the analysis of the security market driven by risk factors, such as economic indicators, social and political instability, and investors′ moods. Therefore, the dependence among asset returns across different investment periods should be assumed. For the MV model in stochastic markets modulated by a Markov chain, Çakmak and Özekici (2006) solve the MV investment problem with only the self-financing constraint and obtain the analytical optimal investment policy. Furthermore, the bankruptcy constraints are added into the MV model in Markovian markets by Wei and Ye (2007), where the analytical optimal investment policy is derived by using the dynamic programming method. The dynamic portfolio optimization problem under multi-factor model in stochastic markets is recently investigated by Chen and Song (2012). Moreover, multi-period investment problems under utility maximization are studied by Çelikyurt and Özekici (2007) and Çanakoğlu and Özekici (2009). The optimal portfolio selection problem with a HARA utility objective function in a continuous-time Markov stochastic market is studied by Çanakoğlu and Özekici (2012). With a self-financing constraint, the optimal investment policy and the corresponding value function are analytically derived. In these papers, a Markov chain is used to reflect abrupt changes in the security prices, typical economic indexes, social instability, and investors′ behavior.

Another way of characterizing the dynamics of stochastic markets is the regime switching approach proposed by Hamilton (1989). The key idea of a regime switching model is to resolve the issue of unobserved economic regimes over time. Asset prices may exhibit different risk levels in different economic situations and, therefore, different risk premiums. For example, there is no clear cut as to whether the economic market is in recession or expansion by directly observing the market data. With a regime switching model, investors can apply a Bayesian analysis to update the probability distribution of regimes with the information on asset returns over time. Many researchers have applied this verstile approach to modeling the complex stock market and obtained sound and reasonable investment strategies. With a self-financing constraint, Zhou and Yin (2003) study the continuous-time MV model under the regime switching setting, while Yin and Zhou (2003) consider a discrete-time analog. Costa and Araujo (2008) study a generalized MV portfolio selection problem with bankruptcy constraints under a continuous-time Markov regime switching market and provide necessary and sufficient conditions for obtaining an optimal control policy. The dynamic optimal consumption and portfolio choice with power utility functions are investigated by Honda (2003) in a continuous-time Markov regime switching market setting. Sotomayor and Cadenillas (2009) consider a consumption–investment problem maximizing the expected total discounted