Insurance: Mathematics and Economics 62 (2015) 202–214

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Nash equilibrium strategies for a defined contribution pension management

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HIGHLIGHTS

- A time-consistent defined contribution pension scheme is considered.
- The problem is formulated as a bi-objective stochastic problem of mean–variance.
- The inflation risk and the salary risk are taken into account.
- Analytical expressions for the time-consistent strategies and value function are obtained.
- Interesting properties of the time-consistent results under inflation are analyzed in detail.

ARTICLE INFO

Article history:
Received March 2014
Received in revised form
March 2015
Accepted 21 March 2015
Available online 1 April 2015

Keywords:
Defined contribution
Stochastic inflation
Stochastic salary
Nash equilibrium strategy
Mean–variance

ABSTRACT

This paper studies the time-consistent investment strategy for a defined contribution (DC) pension plan under the mean–variance criterion. Since the time horizon of a pension fund management problem is relatively long, two background risks are taken into account: the inflation risk and the salary risk. Meanwhile, there are a risk-free asset, a stock and an inflation-indexed bond available in the financial market. The extended Hamilton–Jacobi–Bellman (HJB for short) equation of the equilibrium value function and the verification theorem corresponding to our problem are presented. The closed-form time-consistent investment strategy and the equilibrium efficient frontier are obtained by stochastic control technique. The effects of the inflation and stochastic income on the equilibrium strategy and the equilibrium efficient frontier are illustrated by mathematical and numerical analysis. Finally, we compare in detail the time-consistent results in our paper with the pre-commitment one and find the distinct properties of these two results.

1. Introduction

Nowadays it is well-known that the defined contribution plans have received more attention than the defined benefit plans and have become the dominant form of pension plans all over the world in recent years. The reason for this phenomenon lies in the recognized fact that the aging population problem is threatening the sustainability of a sufficient retirement income for the pensioners, and the defined contribution plan has an advantage over the defined benefit one to ease the press of public financial system by transferring the investment risk from the sponsor to the retiree. In a defined contribution plan, the plan member continuously contributes a fixed proportion of his stochastic salary to the pension fund, and then the contributions are invested in some suitable assets available in the market. Only the contributions to the account are guaranteed while the future benefits fluctuate on the basis of investment earning. Moreover, the DC pension fund investment management usually considers a long time horizon, and the benefits will be obtained nearly on retirement. Therefore, usually one will take broad categories of risks into account, such as the investment risk, the stochastic salary risk and the inflation risk. In the past years, the optimal investment problem in DC pension fund management has been extensively studied. However, an overwhelming amount of literature has focused on maximizing the expected utility of the terminal wealth under the CRRA or CARA criterion. See, for example, Boulier et al. (2001), Battocchio and Menoncin (2004), Cairns et al. (2006), Zhang et al. (2007), Gao

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http://dx.doi.org/10.1016/j.insmatheco.2015.03.014

(2008), Zhang and Ewald (2010), Korn et al. (2011), Di Giacinto et al. (2011), Di Giacinto and Vigna (2012), Han and Hung (2012), and Blake et al. (2013).

As mentioned above, in the past years, most of the existing DC pension fund management problems are based on the expected utility maximization, while there is a delay in solving the relevant problem under mean–variance framework. One reason for this delay, as Vigna (2014) said, lies in the fact that the multi-period and continuous-time versions of mean–variance problem have been produced only quite recently. But since the pioneering work of Markowitz (1952), the mean–variance analysis framework has been the foundation for modern portfolio selection theory; therefore, there appear several investigations on the DC pension fund management under the mean–variance criterion in recent years. Højgaard and Vigna (2007) solve a mean–variance DC management problem for the two-asset case as well as $n+1$-asset case. They compare the mean–variance approach with the target-based approach and prove that the target-based approach can be formulated as a mean–variance problem. Nkeki (2012) considers a mean–variance portfolio selection problem with inflation for a DC pension scheme. Nkeki (2013) examines a mean–variance DC pension management problem with time-dependent salary. He compares the optimal portfolios under the quadratic utility function, the power utility function and exponential utility function. He and Liang (2013) study the optimal control problem of the DC pension plan with the return of premiums clauses in order to protect the rights of the plan members who die early during the accumulation phase. In the financial market only with multiple risky assets, Yao et al. (2013) consider the optimal DC pension fund management with inflation under the mean–variance criterion. They derive the explicit optimal strategy and the efficient frontier through the dynamic programming method and the Lagrange dual theory. Vigna (2014) compares the mean–variance efficient portfolio and the optimal portfolio according to the expected utility, and proves that the CARA and CRRA optimal portfolios are not mean–variance efficient.

Now it is clear that the DC management under mean–variance criterion is a relatively new thing. The objective of the existing relevant papers, however, is seeking an optimal strategy based on the initial information which maximizes the mean–variance utility of the terminal wealth. But it is well known that the dynamic mean–variance criterion lacks of iterated-expectation property, which gives rise to time-inconsistent investment strategy in the sense that the Bellman optimality principle does not hold any more. Here the so-called time-inconsistent strategy means that the optimal strategy obtained at time n does not agree with the optimal strategy derived at time m where $m > n$. Therefore, the optimal strategy in the classical mean–variance model is called as pre-commitment strategy. Recently, this pre-commitment strategy has been criticized for lacking of rationality considering that the investment psychology and taste will change over time, specially when the investment time horizon is relatively long. Thus the decision-maker at a later time might not commit to following a strategy which is not optimal at her current time. For this reason, recent years have seen an upsurge of interest in studying the time-consistent strategy for the mean–variance problem in game theoretic terms. The game theoretic approach to addressing general time inconsistence via Nash equilibrium points can be traced back to Strotz (1956), who proposes three different ways which the decision-maker may follow: (a) The pre-committed decision-maker does not revise her initial strategy even if her strategy is time-inconsistent. She commits her successors to implementing the strategy devised by herself; (b) The naive decision-maker revises her strategy without considering the future revisions even if her strategy is time-inconsistent; (c) The sophisticated decision-maker takes possible future revisions into account, and looks for the (subgame perfect Nash) equilibrium strategy, which is time-consistent. Here we focus on the sophisticated decision-maker in order to obtain the equilibrium strategy for the DC pension management. The entire problem is viewed as a non-cooperative game with one decision-maker at each time n, referred to as decision-maker n ($n = 0, 1, \ldots, T-1$) with T denoting the predefined investment horizon. Given the equilibrium strategies adopted by all the successors, the decision-maker n will choose a decision at her current time to optimize the objective function.

Recently there has been a great interest in dealing with the time inconsistency problem under the mean–variance criterion within a game theoretic framework. For example, Björk and Murgoci (2010) give general approaches to handle time-inconsistent problems by viewing them as a game theoretic framework. They formally define the equilibrium strategy, and derive the extension of HJB equation and its verification theorem for a very general objective function. Zeng and Li (2011) and Li et al. (2012) investigate the mean–variance time-consistent investment and reinsurance policies for an insurer under continuous-time setting. Wu (2013) studies the equilibrium strategy for the multi-period mean–variance portfolio selection problem, and compares in detail the difference between the time-consistent investment strategy and the classical pre-commitment one. Björk et al. (2014) consider a continuous-time mean–variance portfolio optimization problem under the assumption that the risk aversion factor depends dynamically on the current wealth. For more information about time-consistent strategy for mean–variance model, the interested readers are referred to Basak and Chabakauri (2010) and Wang and Forsyth (2011).

Since the DC pension fund management usually lasts for a long time horizon, say 20–30 years, the investment preference and psychology of the decision-makers will inevitably change over time. Consequently, when the strategy lacks of time-consistency, the decision-maker at later time may not commit herself to following the pre-commitment strategy which is not optimal at her current time. In context of the DC pension fund management, no existing literature has involved the Nash equilibrium strategy in closed form under the mean–variance criterion. Our research aims to fill this blank. To this end, we study the Nash equilibrium investment strategy for a defined contribution plan from a theoretical point of view. Meanwhile, we take into account two background risk: the inflation risk and salary risk, and assume that there are one risk-free asset, one stock and one inflation-indexed bond available in the market.

The reminder of this paper is organized as follows. Section 2 formulates the DC pension fund management problem under the mean–variance criterion. In Section 3, the associated extended HJB equation and the verification theory are presented. Section 4 gives the equilibrium strategy and the equilibrium value function. Section 5 analyzes the properties of the equilibrium strategy and the equilibrium efficient frontier. The numerical results are also argued in this section. In Section 6, we compare our results with the case of pre-commitment. Section 7 concludes this paper. Proofs of lemmas and theorems are given in Appendices A–C.

2. Problem formulation

As mentioned in Section 1, since the time horizon of the pension management is usually very long, we cannot neglect the effects of the inflation. For simplicity, the (commodity) price index in this paper is assumed to follow the diffusion process:

$$dI(t) = I(t)\left[\mu_I(t)dt + \sigma_I(t)dW_I(t)\right], \qquad I(0) = I_0, \tag{1}$$

where $\mu_I(t)$ is the instantaneous expected rate of inflation, $\sigma_I(t)$ is the volatility rate of the price index level, and $W_I(t)$ is a standard one-dimension Brownian motion. The price index presents the price for a fixed basket of goods.