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Dynamic lotsizing with a finite production rate

Robert W. Grubbström*

Linköping Institute of Technology, SE-581 83 Linköping, Sweden, Mediterranean Institute for Advanced Studies, Primorski tehnoloski park, SI-5290 Šempeter pri Gorici, Slovenia

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ABSTRACT

The *dynamic lotsizing problem* concerns the determination of optimally produced/delivered batch quantities, when demand, which is to be satisfied, is distributed over time in different amounts at different times. The standard formulation assumes that these batches are provided instantaneously, i.e. that the production rate is infinite.

Using a cumulative geometrical representation for demand and production, it has previously been demonstrated that the *inner-corner condition* for an optimal production plan reduces the number of possible optimal replenishment times to a finite set of given points, at which replenishments can be made. The problem is thereby turned into choosing from a set of zero/one decisions, whether or not to replenish each time there is a demand. If n is the number of demand events, this provides 2^{n-1} alternatives, of which at least one solution must be optimal. This condition applies, whether an Average Cost approach or the Net Present Value principle is applied, and the condition is valid in continuous time, and therefore in discrete time.

In the current paper, the assumption of an infinite production rate is relaxed, and consequences for the inner-corner condition are investigated. It is then shown that the inner-corner condition needs to be modified to a *tangency condition* between cumulative requirements and cumulative production.

Also, we have confirmed the additional restriction for feasibility in the finite production case (provided by Hill, 1997), namely the *production rate restriction*. Furthermore, in the NPV case, one further necessary condition for optimality, *the distance restriction* concerning the proximity between adjacent production intervals, has been derived. In an example this condition has shown to reduce the number of candidate solutions for optimality still further. An algorithm leading to the optimal solution is presented.

1. Introduction and notation

The dynamic lotsizing problem is a variational problem, namely to determine cumulative production over time requiring cumulative production to be at least cumulative requirements at each point in time optimising an objective function. The issue of determining how much to produce (optimal lotsizes) has a history of close to a century (Harris, 1913; Erlenkotter, 1989). Halfway through this history the dynamic lotsizing problem was formulated and solved by a dynamic programming approach (Wagner and Whitin, 1958). Several other algorithms have later been suggested, such as those in Silver and Meal (1973), Federgruen and Tzur (1991), and Grubbström (1999, 2005). For an early overview, see De Bodt et al. (1984).

The majority of approaches have used a discrete time framework in which times (periods), at which requirements appear, are integers. The current author has argued against this restriction for a long time, instead advocating the use of a continuous time framework (of which discrete timing is a special case), and has expressed the dynamic lotsizing problem in continuous time, including continuous-time formulations of the Wagner–Whitin and Silver–Meal algorithms (Grubbström, 1997, 1999, 2005). The *capacitated* dynamic lotsizing problem in continuous time was introduced by Khmelnitsky and Tzur (2004).

The overwhelming volume of literature on the subject of lotsizing is based upon an *Average Cost* (AC) approach, in particular balancing inventory holding costs against fixed setup costs for each batch produced. Against this stands the *Net Present Value* (NPV) principle, based on the monetary stream consequences of decisions (cash flows), compatible with financial methodology. Through the years attention has been given to the differences of these two approaches (Hadley, 1964; Trippi and Levin, 1974; Grubbström, 1980; Kim et al., 1986) and in recent years, an increasing interest in the use of the NPV principle for lotsizing optimisation has appeared (Teunter and van der Laan, 2002; Grubbström and Kingsman, 2004; Beullens and Janssens, 2011). For standard problems, the average cost is shown to correspond to a mixed zeroth and first-order approximation in the interest rate of the NPV expression.

* Tel.: +46 705 127 077.
E-mail address: robert@grubbstrom.com

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It has been recognised for a long time that the dynamic lotsizing problem may be viewed as a binary problem, whether or not to replenish at times when requirements appear (Veinott, 1969). This binary view led to the "inner-corner" condition for a production plan to qualify as a candidate for optimality, and it holds even for complex multi-item production structures (Grubbström et al., 2009; Grubbström and Tang, 2012). The inner-corner condition is a geometrical statement, in the discrete time case equivalent with "Inventory is not to be carried into a period where production takes place" (Aryanezhad, 1992, pp. 425–427).

The only two papers in the literature we have found concentrating on the current problem of dynamic lotsizing with a finite production rate are those of Hill (1997), and of Song and Chan (2005). In the first of these references, the problem is stated exactly as it is here, but in both references the authors restrict the objective function to applying only the AC principle. Neither Hill nor Song and Chan use cumulative functions for requirements and production, and, at least not explicitly, they do not formulate the decision variables in binary form. Because cumulate functions are not used, the inner-corner condition cannot be recognised geometrically. In Hill, however, what is below called the *production rate restriction* is formulated in a similar way, and the main theorem offered by Hill is the same result as ours, but its validity is limited to the AC objective assumption. While Hill formulates and solves the problem in a continuous time framework, Song and Chan start with continuous time, but later split time into discrete periods. But Song and Chan generalise the problem of Hill to a case when AC includes backlogging costs, which is not considered here. Both of these papers end up proposing a dynamic programming algorithm.

The original dynamic lotsizing problem assumes that replenishments take place instantaneously, irrespective of their size. In the current paper the goal is to relax this restriction and study the consequences for the inner-corner condition when the production rate is finite.

The following main notation is introduced (additional notation will be used as the need arises).

n Given number of requirement events.
t_i Given time at which the ith requirement appears, $i=1, 2,\ldots,n$.
$\tau_i=t_{i+1}-t_i$ Time interval between $(i+1)$st and ith requirement events, $i=1, 2, \ldots, n-1$.
D_i Given size of requirement at time t_i, $i=1, 2,\ldots,n$.
$\overline{D}_i=\sum_{j=1}^{i} D_j$ Cumulative requirements immediately after time t_i.
$\overline{P}_i$ Cumulative production immediately after time t_i.
c Unit production cost (out-payment).
K Setup cost for producing one batch (ordering cost).
h Inventory holding cost per unit and time unit.
ρ Continuous interest rate, per time unit.
q Finite production rate, units per time unit.

The following section provides a summary of the classical dynamic lotsizing problem (infinite production rate) when using binary decision variables. Section 3 extends the theory to the case of a constant finite production rate and includes our main theorem and two corollaries. Here a new result is that for optimality two successive production batches may not differ in time more than by a specified expression of the given parameters. In Section 4 expressions are developed for the objective functions and an algorithm is provided, followed by a section with two numerical examples. We finalise with conclusions and a list of references.

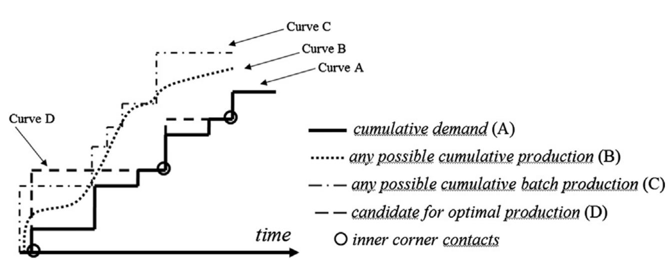

Fig. 1. Illustration of inner-corner condition. From Grubbström and Tang (2012).

2. Brief description of dynamic lotsizing with infinite production rate

In the case of an infinite production rate (treated in essentially all papers hitherto) cumulative demand follows a staircase function and so does cumulative production when the production rate is infinite. A feasible production plan is any staircase function above or which touches the cumulative requirements from above. The basic question is which production staircase to choose when the objective function is to be optimised.

It has been shown earlier that the only solutions that can qualify as candidates for an optimal solution, whether the NPV or AC measure is the objective, are cumulative staircases that fit into inner corners as shown in Fig. 1. This has the consequence that the set of optimal production solutions must be found from those either having an inner-corner contact at a requirement event, or no production. This restricts the number of feasible optimal solutions to 2^{n-1}, since there must always be a contact at the first inner corner (start of process).

Fig. 1 shows the given demand staircase (Curve A, bold) and three types of feasible cumulative production curves (no shortages). Curve B (dotted) illustrates the most general case of feasible production (any non-decreasing curve above or possibly touching cumulative demand), Curve C (dotted and dashed) the general case of feasible production taking place in batches, making cumulative production a staircase function, and Curve D (dashed) a batch-production case which is a candidate for optimality, since it meets the inner-corner condition.

Using the binary variables α_j, with $\alpha_j=1$ meaning a setup at t_j, and $\alpha_j=0$, if not, cumulative production immediately after t_{j-1} will either be $\overline{P}_{j-1}=\overline{D}_{j-1}$ or $\overline{P}_{j-1}=\overline{P}_j$, respectively. As earlier shown, (for instance in Grubbström et al., 2009), this leads to a unique solution

$$\overline{P}_j=\left(\overline{D}_j+\sum_{k=j+1}^{n} D_k \prod_{l=j+1}^{k}(1-\alpha_l)\right), \quad (1)$$

showing how cumulative production depends on the binary decision variables α_i and the given requirements. Here, as in the following, the convention that $\prod_{l=j+1}^{j}(1-\alpha_l)=1$ is adopted. The batch size of production when it takes place (the lotsize), is then given by

$$Q_j=\overline{P}_j-\overline{P}_{j-1}=\alpha_j\sum_{k=j}^{n} D_k \prod_{l=j+1}^{k}(1-\alpha_l) \quad (2)$$

If there is no setup at t_j, i.e. $\alpha_j=0$, then $Q_j=0$.

We may also enquire as to how the timing of supply depends on the decision variables α_j. Let $\overline{T}_j$ denote the time that the supplies immediately after t_j last until. Then, if there is a setup at t_j, then $\alpha_j=1$ and $\overline{T}_{j-1}=t_j$, and if not, $\alpha_j=0$, and $\overline{T}_j=\overline{T}_{j-1}$. This leads to the solution

$$\overline{T}_j=t_{j+1}+\sum_{k=j+1}^{n-1}\tau_k\prod_{l=j+1}^{k}(1-\alpha_l), \quad (3)$$