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Determination of weights for ultimate cross efficiency using Shannon entropy

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ABSTRACT

This paper firstly reviews the cross efficiency evaluation method which is an extension tool of data envelopment analysis (DEA), then we describe the main shortcomings when the ultimate average cross efficiency scores are used to evaluate and rank the decision making units (DMUs). In this paper, we eliminate the assumption of average and utilize the Shannon entropy to determine the weights for ultimate cross efficiency scores, and the procedures are introduced in detail. In the end, an empirical example is illustrated to examine the validity of the proposed method. Some future research directions are also pointed out.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric programming technique for evaluating efficiency of a set of homogenous decision making units (DMUs) with multiple inputs and multiple outputs. It has been proven to be an effective approach in identifying the best practice frontiers and ranking the DMUs. DEA has been extensively applied in performance evaluation and benchmarking of schools, hospitals, bank branches, production plants, and so on (Charnes, Cooper, Lewin, & Seiford, 1994). However, traditional DEA models, such as CCR model in Charnes, Cooper, and Rhodes (1978) can simply classify the DMUs into two groups, namely efficient DMUs and inefficient DMUs. Moreover, it is often possible in traditional DEA models that some inefficient DMUs are in fact better overall performers than some efficient ones. This is because of the unrestricted weight flexibility problem in DEA by being involved in an unreasonable self-rated scheme (Dyson & Thannassoulis, 1988; Wong & Beasley, 1990). The DMU under evaluation heavily weighs few favorable measures and ignores other inputs and outputs in order to maximize its own DEA efficiency.

The cross efficiency method was developed as a DEA extension technique that could be utilized to identify efficient DMUs and to rank DMUs using cross efficiency scores that are linked to all DMUs (Sexton, Silkman, & Hogan, 1986). The main idea of the cross evaluation method is to use DEA in a peer evaluation instead of a self evaluation. There are at least three main advantages for cross-evaluation method. Firstly, it provides a unique ordering among the DMUs (Sexton et al., 1986). Secondly, it eliminates unrealistic weight schemes without requiring the elicitation of weight restrictions from application area experts (Anderson, Hollingsworth, & Inman, 2002). Finally, the cross evaluation method can effectively differentiate between good and poor performers (Boussofiane, Dyson, & Thanassoulis, 1991). Therefore the cross-evaluation method has been widely used for ranking performance of DMUs, for example, efficiency evaluations of nursing homes (Sexton et al., 1986), selection of a flexible manufacturing system (Shang & Sueyoshi, 1995), justification of advanced manufacturing technology (Talluri & Yoon, 2000), diagnosing best intelligent mailer (Kabassi, Virvou, & Despotis, 2003), and so on.

Although average cross efficiency has been widely used, there are still several disadvantages for utilizing the final average cross efficiency to evaluate and rank DMUs, like the losing association with the weights by averaging among the cross efficiencies (Despotis, 2002), which means that this method cannot clearly provide the weights to help decision makers improve their performance, especially, the average cross efficiency measure is not good enough since it is not a Pareto solution. Considering the shortcomings above, Wu, Liang, and Yang (2009) eliminate the average assumption for determining the ultimate cross efficiency scores, and DMUs are considered as the players in a cooperative game, in which the characteristic function values of coalitions are defined to compute the Shapley value of each DMU, and the common weights associate with the imputation of the Shapley values are used to determine the ultimate cross efficiency scores.

In the current paper, we will propose an approach based on information entropy theory instead of calculating the average cross efficiency scores. This approach has several advantages, for example, in this method, the most productive scale size (MPSS) units (Cooper, Seiford, & Tone, 2000) get the best rank and the interior points of the smallest production possibility sets (PPSs) which are inefficient in all models lie at the end of the ranking list (Soleimani & Zarepisheh, 2009). The rest of this paper is organized as fol-

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lows: Section 2 introduces the cross efficiency evaluation method. The new method using Shannon entropy is proposed in Section 3. Section 4 gives an illustrative example, and conclusion and remarks are shown in Section 5.

2. Cross-efficiency evaluation

Using the traditional denotations in DEA, we assume that there are a set of n DMUs, and each DMU_j ($j = 1, 2, \ldots, n$) produces s different outputs using m different inputs which are denoted as x_{ij} ($i = 1, 2, \ldots, m$) and y_{rj} ($r = 1, 2, \ldots, s$), respectively. For any evaluated DMU_d ($d = 1, 2, \ldots, n$), the efficiency score E_{dd} can be calculated by using the following CCR model.

$$\begin{aligned}
\max \sum_{r=1}^{s} \mu_{rd} y_{rd} &= E_{dd} \\
\text{s.t.} \quad \sum_{i=1}^{m} \omega_{id} x_{ij} - \sum_{r=1}^{s} \mu_{rd} y_{rj} &\geqslant 0, \quad j = 1, 2, \ldots, n \\
\sum_{j=1}^{m} \omega_{id} x_{id} &= 1 \\
\omega_{id} &\geqslant 0, \quad i = 1, 2, \ldots, m \\
\mu_{rd} &\geqslant 0, \quad r = 1, 2, \ldots, s.
\end{aligned} \tag{1}$$

For each DMU_d($d = 1, 2, \ldots, n$), we can obtain a group of optimal weights $\omega_{1d}^*, \ldots, \omega_{md}^*, \ \mu_{1d}^*, \ldots, \mu_{sd}^*$ by solving the above model (1), and the cross-efficiency of each DMU_j using the weights of DMU_d, namely E_{dj}, can be calculated as follows.

$$E_{dj} = \frac{\sum_{r=1}^{s} \mu_{rd}^* y_{rj}}{\sum_{i=1}^{m} \omega_{id}^* x_{ij}}, d, \quad j = 1, 2, \ldots, n \tag{2}$$

As shown in the Table 1 of cross efficiency matrix (CEM), for each column, E_{dj} is the cross efficiency score of DMU_j using the weights that DMU_d ($j = 1, 2, \ldots, n$) has chosen. We can also find that the elements in the diagonal are the special cases that can be seen as self-evaluation.

For each DMU_j ($j = 1, 2, \ldots, n$), the average of all E_{dj} ($d = 1, 2, \ldots, n$), namely, $\overline{E}_j = \frac{1}{n}\sum_{d=1}^{n} E_{dj}, \ (j = 1, 2, \ldots, n)$ can be treated as a new efficiency measure, that is, the cross-efficiency score for DMU_j.

3. Determination of ultimate cross efficiency using Shannon entropy

In this section, we will use the Shannon entropy in information theory, which is a rather abstract mathematical concept and can be used as a measure of uncertainty, to determine the ultimate cross efficiency of each DMU. For the detail of Shannon entropy, we will revisit in the following part, which also can be seen in (Soofi & Ehsan, 1990).

Table 1
A generalized cross efficiency matrix.

Rating DMU_d	Rated DMU_j				
	1	2	3	…	n
1	E_{11}	E_{12}	E_{13}	…	E_{1n}
2	E_{21}	E_{22}	E_{23}		E_{2n}
3	E_{31}	E_{32}	E_{33}	…	E_{3n}
…	…	…	…	…	…
…	…	…	…	…	…
…	…	…	…	…	…
n	E_{n1}	E_{n2}	E_{n3}	…	E_{nn}
Mean	$\overline{E}_1$	$\overline{E}_2$	$\overline{E}_3$	…	$\overline{E}_n$

3.1. Entropy and its related knowledge

Information entropy is a measure of uncertainty, which is firstly introduced by Shannon in his paper of *A Mathematical Theory of Communication* (Shannon, 1948), then it has been widely used in many fields such as engineering, management and so on. According to the idea of information entropy, the number or quality of information acquired from decision-making setting is one of the determinants of accuracy and reliability of decision-making problem. Entropy is therefore a very good scale when it is applied to different cases of assessment or evaluation in different decision-making process, and similarly, entropy can also be used to measure the quantity of useful information provided by data itself.

Information entropy is a measurement of uncertainty of the system state, when the system is in limited states with the probability P_i ($i = 1, 2, \ldots, n$) of each state, then the entropy of the system is

$$e = -\sum_{i=1}^{n} P_i \log P_i \tag{3}$$

where

$$0 \leq P_i \leq 1, \quad \sum_{i=1}^{n} P_{i=1} \tag{4}$$

When all the states' probabilities are the same $P_i = \frac{1}{n}(i = 1, 2, \ldots, n)$, the entropy of the system is the maximum, that is

$$e(P_1, P_2, \ldots, P_n) \leq e\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) = \log(n) \tag{5}$$

3.2. Entropy-based weights

According to the concept of entropy, it can be used to evaluate the decision making units (DMUs). Assume there are m alternatives and n evaluation criteria, thus the decision making matrix can be defined as follows:

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

Now, we will introduce the steps for determining the weights of each criterion based on the concept of entropy.

Step 1. *Determination of the closeness*

Firstly, we define the closeness between x_{ij} and its ideal value as d_{ij}, *and* $d_{ij} \in [0, 1]$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

$$d_{ij} = \begin{cases} \dfrac{x_{ij}}{\max\{x_{i1}, x_{i2}, \ldots, x_{in}\}}, & \textit{positive indicators} \\ \dfrac{x_{ij}}{\min\{x_{i1}, x_{i2}, \ldots, x_{in}\}}, & \textit{negtive indicators} \end{cases} \tag{6}$$

Step 2. *Determination of the entropy values for each criterion*

After the above definitions in Step 1, we can define the entropy of the ith criterion as follows:

$$e(d_i) = -k \sum_{j=1}^{n} f_{ij} \ln f_{ij} \tag{7}$$

where $f_{ij} = d_{ij} / \sum_{i=1}^{n} d_{ij}, \ k = 1/\ln n$.

If f_{ij}($j = 1, 2, \ldots, n$) are all the same, the entropy of the ith criterion is the maximum, i.e., $e(d_i) = 1$. And if we assume $f_{ij} = 0$, then $f_{ij} \ln f_{ij} = 0$. Finally, if we normalize $1 - e_i (i = 1, \ldots, n)$, we can obtain the final weights of each criterion (Wang & Lee, 2009).