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A characterization of optimal portfolios under the tail mean–variance criterion

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ABSTRACT

The tail mean–variance model was recently introduced for use in risk management and portfolio choice; it involves a criterion that focuses on the risk of rare but large losses, which is particularly important when losses have heavy-tailed distributions. If returns or losses follow a multivariate elliptical distribution, the use of risk measures that satisfy certain well-known properties is equivalent to risk management in the classical mean–variance framework. The tail mean–variance criterion does not satisfy these properties, however, and the precise optimal solution typically requires the use of numerical methods. We use a convex optimization method and a mean–variance characterization to find an explicit and easily implementable solution for the tail mean–variance model. When a risk-free asset is available, the optimal portfolio is altered in a way that differs from the classical mean–variance setting. A complete solution to the optimal portfolio in the presence of a risk-free asset is also provided.

1. Introduction

In this paper, we consider portfolio selection under the tail mean–variance criterion that was introduced by Landsman (2010):

$$\mathrm{TMV}(L) = \mathbb{E}\left[L \mid L > \mathrm{VaR}_q(L)\right] + \lambda \mathrm{Var}\left[L \mid L > \mathrm{VaR}_q(L)\right]. \quad (1)$$

In the above, (i) $\lambda > 0$, (ii) L is a random loss, with a continuous distribution, on a portfolio, and (iii) $\mathrm{VaR}_q(L)$ is the value-at-risk on the portfolio, defined as

$$\mathrm{VaR}_q(L) = \inf(x \in \mathbb{R} : F_L(x) \geq q), \quad (2)$$

where $q \in (0, 1)$ and $F_L(x)$ is the cumulative distribution function of loss L. ($q = 0.95$ therefore corresponds to a 5% value-at-risk.) The objective is to find an optimal portfolio that minimizes the tail mean–variance criterion subject to a budget constraint.

Since portfolio return $R = -L$, the tail mean–variance criterion may be regarded as an analogue of the classical mean–variance criterion (see e.g., Panjer et al., 1998, p. 379):

$$\mathrm{MV}(L) = \mathbb{E}L + \frac{1}{2}\tau \mathrm{Var}\, L = -\mathbb{E}R + \frac{1}{2}\tau \mathrm{Var}\, R, \quad (3)$$

with $\tau > 0$. This originates from the mean–variance portfolio theory of Markowitz (1952), of course. Unlike its classical counterpart, the tail mean–variance criterion focuses on the behaviour of the tail of portfolio returns through the q-quantile specified in the value-at-risk. This is of interest to portfolio managers whose clients may be concerned with portfolio performance in the event of extreme losses on capital markets.

We make three significant contributions in this paper. First, we use a convex optimization method to find an explicit and easily implementable solution for tail mean–variance optimization. We retain the assumption of joint elliptically distributed returns on securities but, unlike the solution of Landsman (2010), our solution is simple and avoids a sequence of matrix partitions and manipulations. The matrices may be of a large dimension for portfolios containing many hundreds of securities and involving a large variance–covariance matrix, so our solution has a considerable computational advantage. Second, our optimal solution is amenable to a simple interpretation. We provide a simple characterization for the tail mean–variance optimal portfolio in terms of mean–variance efficiency. This facilitates comparison with optimal portfolios under other criteria, such as the mean–variance and value-at-risk criteria. Third, we further extend the work of Landsman (2010) by the inclusion of risk-free lending and borrowing. A complete closed-form optimal solution is provided.

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It is convenient to introduce at this point some notation and assumptions used in the rest of the paper. $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ denote the sets of real numbers, real non-negative numbers, and real positive numbers respectively. We assume that there are n risky securities with mean return $\boldsymbol{\mu} \in \mathbb{R}^n$ and variance–covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. Define $\mathbf{0}$ and $\mathbf{1}$ to be column vectors of zeros and ones respectively, of dimension n. As is usual, we assume that $\boldsymbol{\mu}$ is not collinear with $\mathbf{1}$, i.e., that securities do not all have the same mean return, and that Σ is a (symmetric) positive-definite and non-singular matrix. Let $\mathbf{x} \in \mathbb{R}^n$ be the vector of proportions of wealth invested in a portfolio. Define $\mathcal{P}$ as the set of feasible portfolios of risky securities only:

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{1}^{\mathrm{T}}\mathbf{x} = 1\}. \tag{4}$$

We assume in the above that all wealth is invested. In Section 6, a risk-free asset is included, in which case the proportion of wealth invested in the risk-free asset is $1 - \mathbf{1}^{\mathrm{T}}\mathbf{x}$. There is no constraint other than the budget constraint that all wealth be invested and, in particular, short sales are allowed.

The plan of this paper is as follows. The motivation and literature related to this paper are described in Section 2. In Section 3, we briefly review some results on the tail conditional expectation, the tail variance, and on the tail mean–variance criterion. A characterization of the optimal portfolio under the tail mean–variance criterion using mean–variance efficiency is discussed in Section 4. A novel solution for the tail mean–variance optimal portfolio is derived in Section 5 when portfolios consist of risky securities only. It is illustrated with the help of a numerical example. When risk-free lending and borrowing are introduced, the tail mean–variance optimal portfolio is altered, and this is detailed in Section 6. A few proofs appear in the Appendix.

2. Background and related literature

There are two strands in the development of work related to portfolio optimization. One strand concerns multi-period optimal portfolio selection, initiated by Merton (1969, 1971). Security prices are modelled as continuous-time random processes based on geometric Brownian motion: see Karatzas and Shreve (1991), Yang and Zhang (2005), and Chacko and Viceira (2005), among others. A recent and significant development in this direction is made by Zhao and Rong (2012), who assume investment in multiple risky assets under the constant elasticity of variance (CEV) model with stochastic volatility. This generalizes the geometric Brownian motion model where volatility is deterministic.

In this paper, we choose to follow the second strand of portfolio optimization, namely single-period optimization originally due to Markowitz (1952). There are several reasons why we make this choice. First, the single-period mean-risk model is routinely used in the investment industry because it allows for practical trading constraints, frictional costs, and similar realistic issues. We endeavour in this paper to add to the level of sophistication that may be applied to such models. Second, parameter and model mis-specification risks are very significant in practical portfolio management. The one-period model can accommodate these, for example through a Bayesian approach (see e.g., Garlappi et al., 2007; Tu and Zhou, 2004; Kan and Zhou, 2007), in a way which is readily implementable by practitioners. We supply a closed-form solution in the single-period model which may in future be extended to allow greater robustness. Third, the effect of intertemporal hedging is typically small, so optimal portfolios in the dynamic setting are often (but not always) close to those in the static setting (Chacko and Viceira, 2005). Finally, both individual and institutional investors are concerned with downside and tail risks in portfolio returns, and traditional models fail to capture this. We are able to obtain explicit expressions for optimal portfolios in the one-period setting taking tail risk into account.

In order to solve for tail mean–variance optimal portfolios, we also leverage recent findings on risk measures. Indeed, the tail mean–variance criterion may be viewed as a weighted sum of two risk measures. The first term on the right-hand side of Eq. (1) is the tail conditional expectation of losses, which is identical to the expected shortfall risk measure since losses are assumed to be continuously distributed (see e.g., McNeil et al., 2005, p. 45). Acerbi and Tasche (2002) show that the tail conditional expectation satisfies a set of acceptable properties for risk measures and is therefore deemed to be a coherent risk measure in the sense of Artzner et al. (1999). The second term on the right-hand side of Eq. (1) represents the tail variance, proposed by Furman and Landsman (2006).

Notice that the tail conditional expectation is the best estimate, in a least squares sense, of the worst losses on a portfolio, when losses larger than the q-quantile are considered:

$$\mathbb{E}\left[L \mid L > \mathrm{VaR}_q(L)\right] = \arg\inf_w \mathbb{E}\left[(L-w)^2 \mid L > \mathrm{VaR}_q(L)\right]. \tag{5}$$

On the other hand, the tail variance of Furman and Landsman (2006) gives the estimated squared deviation of the worst losses from the tail conditional expectation:

$$\mathrm{Var}\left[L \mid L > \mathrm{VaR}_q(L)\right] = \inf_w \mathbb{E}\left[(L-w)^2 \mid L > \mathrm{VaR}_q(L)\right]. \tag{6}$$

The tail conditional expectation is explicitly calculated for multivariate distributions that are normal, elliptical, gamma, and Pareto, and for exponential dispersion models by Panjer (2002), Landsman and Valdez (2003), Furman and Landsman (2005), Chiragiev and Landsman (2007), and Landsman and Valdez (2005), respectively. Furman and Landsman (2006) calculate the tail variance in the case of multivariate normal distributions and, more generally, elliptical distributions.

3. Results on tail conditional expectation and tail variance

We start by briefly reviewing some known results on spherical and elliptical distributions and their application to risk management.

If $\mathbf{z} : \Omega \to \mathbb{R}^n$ is spherically distributed with characteristic generator ψ, then its characteristic function is a function of the Euclidean norm of $\mathbf{t}$, where $\mathbf{t} \in \mathbb{R}^n$ is the argument of the characteristic function:

$$\varphi_{\mathbf{z}}(\mathbf{t}) = \mathbb{E}\left[\exp\left(i\mathbf{t}^{\mathrm{T}}\mathbf{z}\right)\right] = \psi\left(\frac{1}{2}\|\mathbf{t}\|^2\right). \tag{7}$$

ψ is termed the characteristic generator because it specifies different members of the spherical family of distributions. For example, the standard normal random variable has characteristic function $\exp\left(-t^2/2\right)$.

Elliptical distributions are affine transformations of spherical distributions, so the probability density contours are distorted from spheroids to ellipsoids. Let $\mathbf{r} : \Omega \to \mathbb{R}^n$ be a vector of asset returns on n securities. If $\mathbf{r}$ is elliptically distributed with location vector $\boldsymbol{\mu} \in \mathbb{R}^n$, dispersion matrix $\Sigma \in \mathbb{R}^{n\times n}$, and characteristic generator ψ, then its characteristic function is

$$\varphi_{\mathbf{r}}(\mathbf{t}) = \mathbb{E}\left[\exp\left(i\mathbf{t}^{\mathrm{T}}\mathbf{r}\right)\right] = \exp\left(i\mathbf{t}^{\mathrm{T}}\boldsymbol{\mu}\right)\, \psi\left(\frac{1}{2}\mathbf{t}^{\mathrm{T}}\Sigma\mathbf{t}\right). \tag{8}$$

For further details on spherical and elliptical distributions, see Fang et al. (1990).

A key property of elliptically distributed random variables with the same characteristic generator ψ is that any linear combination of these random variables is also elliptically distributed with