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Constructive and *computable* Hahn–Banach theorems for the (second) fundamental theorem of welfare economics☆

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ABSTRACT

The Hahn–Banach Theorem plays a crucial role in the second fundamental theorem of welfare economics. To date, all mathematical economics and advanced general equilibrium textbooks concentrate on using non-constructive or incomputable versions of this celebrated theorem. In this paper we argue for the introduction of constructive or computable Hahn–Banach theorems in mathematical economics and advanced general equilibrium theory. The suggested modification would make applied and policy-oriented economics intrinsically computational.

1. A preamble

> "There are *many* – *denumerably*, I suspect – other versions of the [Hahn–Banach] theorem, ... It has many applications not only outside functional analysis but outside mathematics".
> Narici (2007, p. 88); italics added.

The three 'crown jewels' of the mathematical economics of the second half of the twentieth century are undoubtedly the proof of the *existence* of a *Walrasian Exchange Equilibrium* and the mathematically rigorous demonstration of the validity of the *two fundamental theorems of welfare economics*.[1] Brouwer's original fix-point theorem – and a variety of its extensions, primarily that of Kakutani – was instrumental in the classic Arrow–Debreu proof of the former. A variety of variations of the classic *Hahn–Banach theorem* (henceforth referred to as *H–B T*) was used in the formal proof of the latter—again, one could, with considerable doctrine historical legitimacy[2] attribute the use of variants of this mathematical gem to Arrow (1951) and Debreu (1954; 1984, p. 269).

Even within these three 'crown jewels', it is arguably the second fundamental theorem of welfare economics (henceforth referred to as *FTWE II*), that is most fundamental from the point of view of the applied, policy-oriented, economist. For, as Starr (2011, p. 213; italics added) states:

> "The Second Fundamental Theorem of Welfare Economics represents *a significant defense of the market economy's resource allocation mechanism*".

However, every appeal to, or statement of, the H–B T, or its various implications – the most common being in its supporting or separating hyperplane forms – in any of the standard textbooks of general equilibrium theory or mathematical economics,[3] are to its *non-constructive* or *incomputable* versions (Arrow and Hahn, 1971, p. 382, Stokey et al., 1989, pp. 45–451, Bridges, 1998, Chapter 6 and

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1 It is appropriate also that two of our neoclassical founding fathers are associated with the names commonly assigned to these 'crown jewels': *Léon Walras* and *Vilfredo Pareto*.

2 But see Blaug (2007).

3 Obviously the caveat '*to the best of this author's knowledge*' is implicitly invoked here.

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Appendix C).[4] In every one of these statements of the H–B T, there is an appeal to one or another non-constructive or incomputable precept. For example, in the Stokey–Lucas allegedly *Geometric form of the Hahn–Banach Theorem*[5] (*ibid*, 450), an appeal is (implicitly) made to Zorn's Lemma and transfinite induction, as does Bridges (p. 263). The caveat, in the Arrow–Hahn use of the H–B T, of its validity in *finite-dimensional spaces* (ibid, p. 382), may *deceive* the unwary reader into thinking that finite counting arguments may obviate the constructive or computable requirements. That this is not so can be discerned from a careful perusal of Schechter (1971, Chapter II, Section 2).

The rest of this paper is structured as follows. In the next section the ε-approximate constructive H–B T is stated and the way it may be used in economics is outlined. A similar task is undertaken for the computable H–B T, as much as possible paralleling Bishop's ε-approximate constructive H–B T.

In the concluding Section 3 a brief outline of Ishihara's version of the exact – i.e., dispensing with the ε-approximation – H–B T is outlined. However, I am not convinced the trade-off for the exact H–B T, in terms of gain in economic intuition, is sufficiently significant from any applicable, policy-oriented point of view for it to be adopted in economic analysis.

Some notes on the ε-approximate, intuitionistic–constructive Brouwer fix-point theorem are also added in the concluding section.

2. Constructive and computable Hahn–Banach Theorems

> "Throughout [Computability in Analysis and Physics] we have attempted to give general principles from which the effectivization or noneffectiveness of well-known classical theorems follow as corollaries .... For example, it would be interesting to have a general principle which gave as corollary the known facts concerning the Hahn–Banach Theorem". Pour-El and Richards (1989, p. 194).

Four constructive and computable versions of the H–B T are now given. The first two, are referred to as Theorems B and I; the next two referred to as M-N1 and M-N2, respectively.[6]

Theorem B (*Bishop, 1967, Theorem 4, p. 263; italics added*)**.** *Let λ be a nonzero linear functional on a linear subset V of a* separable *normed linear space B, whose null space – i.e.,* kernel *– $N(\lambda)$ is a* located *subset of B. Then for each $\varepsilon > 0$ there* exists *a* normable *linear functional ν on B with $\nu(x) = \lambda(x)$ for all x in V, and $\|\nu\| \leq \|\lambda\| + \varepsilon$.*

Bishop constructs *Brouwerian counterexamples*[7] to demonstrate the necessity of the ε. A particularly illuminating hint for the construction of a relevant Brouwerian counterexample to show the necessity of the ε in Bishop's constructive H–B T is given in Bridges and Richman (1987, p. 46, Problem 19).

Any mathematically trained economist, with a solid grounding in the mathematics of advanced general equilibrium theory would have no difficulty in coming to terms with almost all the concepts used in Theorem B—except *located* and *normable* (perhaps). Furthermore no applied economist, with grounding in mathematical general equilibrium theory would have any reason to accept a *separable* normed linear space. However, it is not at all clear that any standard course in mathematical economics or advanced general equilibrium theory – using one or another of the textbooks cited in the previous section or, indeed, any other – would read '*there exists*' in the above statement of the constructive H–B T in its *constructive sense*.

To be constructively precise, the ordinary operations of *scalar multiplication, vector addition* and *norm* have to be *constructively determined*. A *located* subset $N(\lambda)$ of B means it is possible to *constructively calculate* the distance, say $d(x)$, of any point $x \in B$, from $N(\lambda)$; and, a *normable* linear functional (Bridges and Richman, 1987, p. 249), λ, is defined as one for which $\|\lambda\|$ *exists*, where:

$$\|\lambda\| \equiv \text{l.u.b}\{|\lambda(x)| : x \in S\}.$$

Thus, the *constructive* meaning of Theorem B is the following:

Remark 1. Given the constructive real number ε, it is possible to constructively calculate ν, and its norm, from a constructive representation of λ, the constructively determined norm of λ, constructively defined vector addition and scalar multiplication in B, with its constructive norm, $\|\cdot\|$.

Ishihara (1989), by restricting the separable normed linear space, showed how Bishop's ε-approximate constructive H–B T can be replaced by one in which the constructive real number ε could be dispensed with. In the restriction, Ishihara uses the notion of a *Gâteaux differentiable* norm and *uniform convexity*.[8]

Theorem I (*Ishihara, 1989, p. 80; italics added*)**.** *Let M be a linear subset of a* uniformly convex *complete normed linear space E with a* Gâteaux differentiable norm *and let f be a nonzero normable linear functional on M. Then there exists a unique linear functional g on E such that $g(x) = f(x)$ for all x in M and $\|g\| = \|f\|$.*

It goes without saying that *all* of the technical concepts used in the statement of Theorem I are *constructively defined*, in particular the phrase '*there exists*'. However, it does not seem to me to be reasonable to expect an applied or a policy-oriented economist would be persuaded that the gains from introducing new, albeit constructively defined, notions outweigh the intuitive acceptability of the constructive underpinnings of Bishop's ε-approximate H–B T.

Turning next to computable Hahn–Banach Theorems, there are two, almost exactly parallel results, to the above two by Bishop and Ishihara for *constructive analysis*, developed by Metakides, Nerode (*op.cit*), for *computable analysis*.

4 It will be noted that I have chosen the more standard references, but all of them published *after* Bishop (1967), where a crystal clear constructive version of the H–B T was first given (ibid, pp. 262–263). However, there were, of course, earlier economic classics – for e.g., Debreu (1959, Chapter 6) and Malinvaud (1953, p. 245) – invoking some variant of the H–B T. There is, surely, no excuse for anyone to invoke a *non-constructive* H–B T after 1967 (and to an *incomputable* H–B T after the appearance of Metakides and Nerode, 1982).

5 Clearly, Stokey–Lucas statement of the H–B T is adapted from Luenberger (1969, Section 5.11, particularly p. 133), where it is (correctly) attributed to Mazur (1933), without however any specific reference. It was first referred to as the '*Geometric* Hahn–Banach Theorem' by Bourbaki (cf., Narici, *op.cit*, p. 88), whose notion of 'geometry' was singularly non-intuitive, contrary to Luenberger's attempted justification for the name on intuitive grounds. This is not irrelevant in the context of Bishop's constructive version of the H–B T, using also *Brouwerian counterexamples* (see, for example, Mandelkern, 1989), in a crucial way. Luenberger cheerfully acknowledges the use and invoking of Zorn's Lemma, which is not even mentioned in Stokey–Lucas (op.cit).

6 Referring to the Bishop (B) and Ishihara (I) H–B T results in constructive analysis; M-N1 and M-N2 refer to Metakides and Nerode (1982, 1985) theorems on H–B T, within the framework of computable analysis.

7 See, again, Mandelkern (1989), especially p. 3 (bold italics, added):

"Brouwer's critique of... [classical mathematics showed that] certain classical theorems could not be true, for if they were true, many... would be trying to use them to ***solve the unsolved*** problems. Such a demonstration is called a *Brouwerian counterexample*; it differs from an ordinary counterexample, which demonstrates that a given statement implies another statement which is known to be false".

8 See Johns and Gibson (1981), where these concepts were first used in the constructive analysis of duality in Orlicz Spaces.