Insurance: Mathematics and Economics 51 (2012) 26–42

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal dividend policies for compound Poisson processes: The case of bounded dividend rates

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A R T I C L E   I N F O

Article history:
Received June 2011
Received in revised form
February 2012
Accepted 27 February 2012

JEL classification:
G22
G35

Keywords:
Cramér–Lundberg process
Insurance
Bounded dividend rates
Optimal investment policy
Hamilton–Jacobi–Bellman equation
Viscosity solution
Risk control
Threshold strategy
Band strategy

A B S T R A C T

We consider in this paper the optimal dividend problem for an insurance company whose uncontrolled reserve process evolves as a classical Cramér–Lundberg model with arbitrary claim-size distribution. Our objective is to find the dividend payment policy which maximizes the cumulative expected discounted dividend pay-outs until the time of bankruptcy imposing a ceiling on the dividend rates. We characterize the optimal value function as the unique bounded viscosity solution of the associated Hamilton–Jacobi–Bellman equation. We prove that there exists an optimal dividend strategy and that this strategy is stationary with a band structure. We study the regularity of the optimal value function. We find a characterization result to check optimality even in the case where the optimal value function is not differentiable. We construct examples where the claim-size distribution is smooth but the optimal dividend policy is not threshold and the optimal value function is not differentiable. We study the survival probability of the company under the optimal dividend policy. We also present examples where the optimal dividend policy has infinitely many bands even in the case that the claim-size distribution has a bounded density.

1. Introduction

The seminal paper of De Finetti (1957) presented the problem of maximizing the expected discounted dividend payments until the possible ruin of a company. From then on, there has been a lot of research on this issue. We refer to Avanzi (2009), Albrecher and Thonhauser (2009) and the book of Schmidli (2008) as surveys on this issue.

We consider the optimal dividend problem for an insurance company whose uncontrolled reserve process evolves as a classical Cramér–Lundberg model. Our objective is to find the dividend payment policy which maximizes the cumulative expected discounted dividend pay-outs until the time of ruin, imposing a ceiling on dividend rates. This problem was addressed in the paper of Gerber and Shiu (2006) and in Section 2.4 of the book of Schmidli (2008). Gerber and Shiu (2006) gave a verification result for optimality, namely, when the value function of a given admissible dividend strategy is smooth and satisfies the Hamilton–Jacobi–Bellman equation then it is the optimal value function. They found the value function of any threshold strategy and showed that the optimal strategy is threshold in the case that the claim-size distribution density is a mixture of exponential densities. Schmidli (2008) considered the case where the claim-size distribution is smooth. He studied regularity properties of the optimal value function and gave a more general verification theorem for optimality.

In this paper we allow any claim-size distribution. We study the regularity of the optimal value function and we prove that the optimal value function, regardless of its smoothness, can be fully characterized as the unique bounded viscosity solution of the associated HJB equation. Let us point out that in the case of unbounded dividends the optimal value function is characterized as the smallest viscosity solution of the corresponding HJB equation (see Azcue and Muler (2005)). This result still holds in the case of bounded dividends but the characterization result is simpler because there is only one bounded viscosity solution. In the unbounded case the optimal value functions were not bounded and had a linear growth at infinity.

The main result of this paper is to show that there exists an optimal dividend strategy and to find its structure. Since the optimal strategies are not always threshold (even if the claim-size distribution function is smooth), it is necessary to introduce a larger class of strategies called band strategies. A band strategy is characterized by three sets which partition the state space of the surplus process. Each set is associated with a certain

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doi:10.1016/j.insmatheco.2012.02.011

dividend payment action: either paying the incoming premium as dividends, or paying no dividends or paying dividends at the maximum rate allowable. The topological properties of these sets depend on the relation between the ceiling and the premium rate. When the ceiling rate is greater than or equal to the premium rate, the band strategies are similar to the ones defined in Gerber (1969) and Azcue and Muler (2005) where the same optimization problem but with unbounded dividends was studied. In the case that the ceiling rate is smaller than the premium rate, we need to define another class of band strategies because all the controlled surplus trajectories are piecewise increasing and it is not possible to pay the incoming premium as dividends. We can find, via a fixed-point argument, the value function of the band strategies in both cases. To show that there exists an optimal dividend strategy, we use the fact that the optimal value function satisfies the HJB equation in order to construct a partition of the state space of the surplus process and prove that the value function of the band strategy associated with this partition coincides with the optimal value function. A matter of future research would be to find sufficient conditions on the claim-size distribution to ensure that the optimal strategy is threshold.

We find examples where the optimal dividend strategy has infinitely many bands even in the case that the claim-size distribution has bounded density; in these examples the ceiling rate is equal to the premium rate.

We derive a method for finding the optimal dividend strategy when this strategy has a finite band structure. First we show how to obtain the value function of any finite band dividend strategy (these value functions could be non-continuous when the ceiling rate is greater or equal the premium rate and the band dividend strategy is not threshold). Then, we look for the best one-band strategy (threshold strategy) and check whether the value function of this strategy is the optimal one. If this is not the case, we construct the best two-band strategy and check if it is the optimal one and so on.

Using this method we show examples where the optimal dividend strategy is not threshold when the ceiling rate is less, equal and greater to the premium rate; in these examples a gamma claim-size distribution is used. We also present an example where the optimal strategy is threshold but the optimal value function is not smooth at the threshold level, in this case the claim size is constant.

Finally, we study the survival probability of the company under the optimal dividend strategy.

In Section 2, we give a precise formulation of the optimization problem. In Section 3, we prove basic properties of the optimal value function and show that it is a viscosity solution of the corresponding HJB equation. In Section 4, we characterize the optimal value function as the unique bounded viscosity solution of the HJB equation. In Section 5, we define band strategies and characterize their value functions. In Section 6, we prove that there exists an optimal dividend strategy and show that this strategy is stationary and has a band structure. In Section 7, we show how to construct the optimal value function in the case that the optimal strategy has a finite band structure and give some examples. Finally, in Section 8, we study the survival probability of the company under the optimal dividend strategy.

2. Statement of the problem

We assume that the uncontrolled surplus process of an insurance company follows the classical Cramér–Lundberg process

$$S_t = S_0 + pt - \sum_{i=1}^{N_t} U_i,$$

where S_0 is the initial surplus, $p > 0$ is the constant premium intensity, N_t is the number of claims up to time t and U_i is the size of the i-th claim. Let us call τ_i the arrival time of the i-th claim. The random variables U_i are i.i.d. with distribution function F and $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\beta > 0$ independent of the U_i's. We define Ω as the set of càdlàg paths and $(\Omega, \mathcal{F}, P)$ as the complete probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the processes S_t.

A *control strategy* is a process $\pi = (L_t)_{t\geq 0}$ where L_t is the cumulative dividends the company has paid out until time t. Given the dividend-rate ceiling l_0, we say that the control strategy π is *admissible* if the process L_t is predictable, non-decreasing, continuous and satisfies

$$L_{t+h} - L_t \leq l_0 h \quad \text{for } h \geq 0.$$

Denote by Π_x the set of all the admissible control strategies with initial surplus x. For any $\pi \in \Pi_x$, *the controlled risk process* X_t^π can be written as

$$X_t^\pi = x + pt - \sum_{i=1}^{N_t} U_i - L_t. \tag{2.1}$$

We say that $\tau^\pi = \inf\{t \geq 0 : X_t^\pi < 0\}$ is the *ruin time* of the company, note that it can only occur at the arrival of a claim. We define the value function of the control strategy π by

$$V_\pi(x) = E_x\left(\int_0^{\tau^\pi} e^{-cs} dL_s\right), \tag{2.2}$$

where $c > 0$ is the discount factor. The integral is interpreted pathwise in a Riemann–Stieltjes sense.

We want to solve the following optimization problem

$$V(x) = \sup\{V_\pi(x) \text{ with } \pi \in \Pi_x\} \quad \text{for } x \geq 0. \tag{2.3}$$

For technical reasons, we define $V(x) = 0$ for $x < 0$.

3. Hamilton–Jacobi–Bellman equation

In this section we prove basic properties of the optimal value function and show that it is a viscosity solution of the corresponding HJB equation.

The next proposition states basic properties of the optimal value function. It is proved in Lemma 2.31 in Schmidli (2008).

Proposition 3.1. *The optimal value function V is continuous, increasing, positive with limit l_0/c at infinity. Moreover V is Lipschitz with constant $K = l_0(c+\beta)/(cp)$.*

Let us define

$$\mathcal{D} = \{x \geq 0 : V \text{ is differentiable and } F \text{ is continuous at } x\}. \tag{3.1}$$

Since V is Lipschitz, it is absolutely continuous and since the discontinuities of the claim-size distribution function F are countable, the set $\mathcal{D}$ has full measure.

The next proposition states the dynamic programming principle (DPP) of the optimal value function. The proof is analogous to the one of Proposition 3.1 of Azcue and Muler (2005).

Proposition 3.2. *For any $x \geq 0$ and any stopping time τ, we can write*

$$V(x) = \sup_{\pi=(L_t)\in\Pi_x} E_x\left(\int_0^{\tau\wedge\tau^\pi} e^{-cs} dL_s + e^{-c(\tau\wedge\tau^\pi)} V(X^\pi_{\tau\wedge\tau^\pi})\right).$$

We now obtain heuristically the HJB equation. We assume a priori that V is continuously differentiable at x and that there