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A category approach to relation preserving functions in rough set theory ☆

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ABSTRACT

The category **Rel** whose objects are all pairs (U, r), where r is a relation on a universe U, and whose morphisms are relation-preserving mappings is a canonical example in category theory. One of the convenient categories for rough set systems on a single universe is **Rel** since the objects of **Rel** are approximation spaces. The morphisms of a ground category **dfTex** whose objects are textures can be characterized by definability. Therefore, we particularly investigate a textural counterpart of the category **Rel** denoted by **diRel** of textural approximation spaces and direlation preserving difunctions. In this respect, we prove that **diRel** is a topological category over **dfTex** and **Rel** is a full subcategory of **diRel**. In view of the textural arguments, we show that the preimage of a definable subset of an approximation space with respect to a relation preserving function is also definable in the category **Rere** of reflexive relations. Furthermore, we denote the category of all information system homomorphisms and all information systems by **IS** and we show that the category **ISO** of all information system homomorphisms and all object-irreducible information systems where the attribute functions are surjective is embeddable into **Rel**.

0. Introduction

It is well-known that the category **Rel** whose objects are all pairs (U, r), where r is a relation on a universe U, and whose morphisms are relation-preserving mappings is a canonical example in category theory [1]. In rough set theory, the pair (U, r), as an object of **Rel**, is called an approximation space. Recently, Juan Lu et al. studied a different version of this category denoted by M-**IndisSp** whose objects are M-indiscernibility spaces and the morphisms M-equivalence relation-preserving mappings where M is a fixed index set for the families of equivalence relations on a given universe [26]. If M is countable, then M-indiscernibility spaces are multiple-source approximation systems defined by Khan and Banerjee in [23]. Essentially, if the index set M is a singleton, then M-**IndisSp** turns into a category of approximation spaces and equivalence relations which is a full subcategory of **Rel**. In fact, an M-indiscernibility space is a dynamic relational system defined by Pagliani in [27]. Some applications on dynamic relational systems can be found in [22,37,38]. Recently, there have been developments in the subject of textural rough sets [11–15]. A ground category in texture space theory is **dfTex** whose objects are textures and morphisms are difunctions. Difunctions can be characterized using the prime concept of definability of rough set theory. That is, for any two textures $(U, \mathcal{U})$ and $(V, \mathcal{V})$, a direlation $(r, R) : (U, \mathcal{U}) \to (V, \mathcal{V})$ is a morphism in **dfTex** if and only if every subset $A \in \mathcal{U}$ is (r, R)-definable [8]. In the context of rough sets, this result can be stated as a fact that a relation $r : U \to V$ is a function if and only if every subset of V is r-definable. Recall that the presections with respect

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to direlations are natural generalizations of lower and upper approximations based on relations. Hence, from the textural point of view, one can observe that the lower and upper approximations of a set A can be given using the point-free formulations $\underline{app}_r(A) = U \setminus r^{-1}(V \setminus A)$ and $\overline{app}_r(A) = r^{-1}(A)$ for any subset $A \subseteq V$ where $r : U \to V$ is a relation [15]. These arguments help us to focus our attention on a new category denoted by **diRel** of textural approximation spaces and direlation preserving difunctions. One of the important full subcategories of **Rel** is the category of sets and reflexive relations denoted by **Rere** [1]. A textural approach shows that the preimage of a definable subset of an approximation space with respect to a relation preserving function is also definable in the category **Rere**. On the other hand, invariance of upper and lower approximations under information system homomorphisms are studied in [36]. Here, we note that information systems and information system homomorphisms form a category denoted by **IS**. An object function of an information system homomorphism between information systems is a relation preserving function with respect to equivalence relations determined by the attribute sets. Hence, definable sets of information systems can be also considered using the object functions of information system homomorphisms.

Note that the category **REL** of sets and relations is subject to rough set models on two universes and it is isomorphic to the category **R-APR** of power sets and approximation operators [16,30]. Hence, the categories **Rel** and **REL** have different directions. Some works on categorical results in rough set theory can be found in [2,3,16–19,25,26]. For the basic categorical results and terminology, we refer to [1].

This paper is organized as follows. In Section 1, we recall the motivation, and necessary concepts and results related to textures from [6–9,15]. Section 2 is devoted to direlation preserving difunctions. Here, we show that textural approximation spaces and direlations preserving difunctions form a category denoted by **diRel**. We prove that **Rel** is isomorphic to a full subcategory of **diRel**. Further, we show that **diRel** is a topological category and hence, it has products and sums. In Section 3 we discuss textural definability and bicontinuity. In particular, we prove that under pre-images of relation preserving difunctions, textural definability is preserved. Sections 4 and 5 are devoted to approximation spaces and information systems, respectively. First, we give some basic results related to definability and we discuss the category **Rel** and the category **IS** of information systems and information system homomorphisms.

1. Textures

Let U be a set. Then $\mathcal{U} \subseteq \mathcal{P}(U)$ is called a *texturing* of U, and $(U, \mathcal{U})$ is called a *texture space*, or simply a *texture* [6], if

(i) $(\mathcal{U}, \subseteq)$ is a complete lattice containing U and $\emptyset$, which has the property that arbitrary meets coincide with intersections, and finite joins coincide with unions,

(ii) $\mathcal{U}$ is completely distributive, that is, for all index set I, and for all $i \in I$, if J_i is an index set and if $A_i^j \in \mathcal{U}$, then we have

$$\bigcap_{i\in I} \bigvee_{j\in J_i} A_i^j = \bigvee_{\gamma \in \prod_i J_i} \bigcap A^i_{\gamma(i)}.$$

(iii) $\mathcal{U}$ separates the points of U, that is, given $u_1 \neq u_2$ in U there exists $A \in \mathcal{U}$ such that $u_1 \in A$, $u_2 \notin A$, or $u_2 \in A$, $u_1 \notin A$.

A complementation on $(U, \mathcal{U})$ is a mapping $c_U : \mathcal{U} \to \mathcal{U}$ satisfying the conditions

$$\forall A \in \mathcal{U}, \quad c_U^2(A) = A,$$
$$\forall A, B \in \mathcal{U}, \quad A \subseteq B \quad \Rightarrow \quad c_U(B) \subseteq c_U(A).$$

Then the triple $(U, \mathcal{U}, c_U)$ is said to be *a complemented texture space*.

In a texture $(U, \mathcal{U})$, p-sets and q-sets are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\} \quad \text{and} \quad Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\},$$

respectively. The condition (ii), that is, the complete distributivity of $(U, \mathcal{U})$ is equivalent to the following statement [10].

(ii)′ For $A, B \in \mathcal{U}$, if $A \not\subseteq B$ then there exists $u \in U$ with $A \not\subseteq Q_u$ and $P_u \not\subseteq B$.

A nonempty set $A \in \mathcal{U}$ is *a molecule* if $\forall B, C \in \mathcal{U}$, $A \subseteq B \cup C \Rightarrow A \subseteq B$ or $A \subseteq C$. Clearly, p-sets are molecules of a texture space. A texture space $(U, \mathcal{U})$ is called *simple* if all molecules of the space are p-sets.

A trivial example of a texture is the pair $(U, \mathcal{P}(U))$ where $\mathcal{P}(U)$ is the power set of U. It is called *a discrete texture*. Clearly, $(U, \mathcal{P}(U))$ is simple and for $u \in U$ we have

$$P_u = \{u\} \quad \text{and} \quad Q_u = U \setminus \{u\}$$

and $c_U : \mathcal{P}(U) \to \mathcal{P}(U)$ is the ordinary complementation on $(U, \mathcal{P}(U))$ defined by $c_U(A) = U \setminus A$ for all $A \in \mathcal{P}(U)$. However, the basic motivation of textures is in fact, the natural correspondence between the fuzzy lattices and simple textures [7].