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Fuzzy Sets and Systems 276 (2015) 100–113
FUZZY sets and systems
www.elsevier.com/locate/fss

Tense operators in fuzzy logic ☆

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Received 31 March 2013; received in revised form 28 August 2014; accepted 8 September 2014

Available online 16 September 2014

Abstract

The aim of the paper is to introduce and describe tense operators in every fuzzy logic which is axiomatized by means of a residuated poset. For this we use the axiomatization of universal quantifiers as a starting point and we modify these axioms for our sake. At first, we show that the operators can be recognized as modal operators and we study the pairs as the so-called dynamic pairs. Further, we get constructions of these operators in the corresponding residuated poset provided a time frame is given. Moreover, we solve the problem of finding a time frame in the case when the tense operators are given. In particular, any tense algebra is representable in its Dedekind–MacNeille completion.

Keywords: Fuzzy logic; Modal logic; Residuated poset; Tense logic; Tense operators; Fuzzy dynamic algebra

1. Introduction

It is known that propositional logics, both classic or non-classic, do not incorporate the dimension of time. To obtain a tense logic we enrich the given propositional logic by new unary operators which are denoted by G, H, F and P in [3,5,9,11].

The aforementioned operators G, H, F and P are usually called *tense operators*. They are in certain sense quantifiers which quantify over the time dimension of the logic under consideration. The semantical interpretation of these tense operators G and H is as follows. Consider a pair $(T, \leq)$ where T is a non-void set and $\leq$ is a partial order on T. Let $x \in T$ and $f(x)$ be a formula of a given logical calculus. We say that $G(f(t))$ *is valid* if for any $s \geq t$ the

☆ Both authors acknowledge the support by ESF Project CZ.1.07/2.3.00/20.0051 Algebraic methods in Quantum Logic of the Masaryk University.
Supported by european social fund in the czech republic, EUROPEAN UNION, MINISTRY OF EDUCATION, YOUTH AND SPORTS, OP Education for Competitiveness — INVESTMENTS IN EDUCATION DEVELOPMENT .
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http://dx.doi.org/10.1016/j.fss.2014.09.007
0165-0114/

formula $f(s)$ is valid. Analogously, $H(f(t))$ is valid if $f(s)$ is valid for each $s \leq t$. Thus the unary operators G and H constitute an algebraic counterpart of the tense operations "it is always going to be the case that" and "it has always been the case that", respectively. These tense operators were firstly introduced as operators on Boolean algebras (see [3] for a overview).

Analogously, the operators F and P can be considered in certain sense as existential quantifiers "it will at some time be the case that" and "it has at some time been the case that".

It is worth noticing that the operators G and H can be considered as certain kind of modal operators which were already studied for intuitionistic calculus by Wijesekera [21], further by Dellunde, Godo and Marchioni [10] and by Mukherjee and Dasgupta [18], in the de Morgan framework by Cattaneo, Ciucci and Dubois [4], for tense symmetric Heyting algebras by Figallo, Pelaitay and Sanza [14] and in a general setting by Ewald [13]. For the logic of quantum mechanics (see e.g. [12] for details of the so-called quantum structures), the underlying algebraic structure is e.g. an orthomodular lattice or the so-called effect algebra (see [12,15]) and the corresponding tense logic was treated in [6,7,19], in a bit more general setting also in [2].

It is well-known that the semantics of a given fuzzy logic can be formally axiomatized by means of a residuated poset, see [1] for details. Hence, we assume that a corresponding residuated poset is given and its connection to the semantics of a fuzzy logic is known. When introducing tense operators for this logic, we would like to know how to extend this axiomatization to capture also a semantics of these tense operators. Purely algebraically, we are interested in properties and a construction of suitable tense operators in a given residuated poset. In our paper, we will solve these questions algebraically, problems concerning logical interpretation are not a goal of our study.

If the logical calculus under consideration has a negation connective (denoted by $'$) and if it satisfies the double negation law (i.e. $x'' = x$ for every its proposition x) then we can define P and F by means of G and H as follows:

$$P(x) = H\big(x'\big)' \quad \text{and} \quad F(x) = G\big(x'\big)'.$$

However, this is not possible in a fuzzy logic where $x''' = x'$ but x'' need not be equal to x.

Since the tense operators G and H can be considered as universal quantifiers in the aforementioned case, we can try to axiomatize them by means of some of the known axiomatization for universal quantifiers which is as follows:

(U1) $\forall(1) = 1$,
(U2) $\forall(x) \leq x$,
(U3) $\forall(x \wedge y) = \forall(x) \wedge \forall(y)$,
(U4) $\forall(\forall(x)) = \forall(x)$,
(U5) $\forall((\forall(x))') = (\forall(x))'$.

Let us mention that this system of axioms implicitly assumes that the logical calculus in question is ordered (and hence the relation $\leq$ in (U2) is a partial ordering) with the greatest element 1 and it can be considered as a meet-semilattice (where $\wedge$ in (U3) is the infimum).

This is of course satisfied when a residuated lattice is under consideration. However, we would like to study a more general case, namely residuated posets which need not be lattices with respect to the order. Hence, we prefer to substitute the axiom (U3) by a weaker form as follows

$$x \leq y \quad \text{implies} \quad \forall(x) \leq \forall(y).$$

It is a remarkable fact that the above mentioned system of axioms (U1)–(U5) does not consider other logical connectives except negation and the connective $\wedge$ (which need not be a conjunction, see [1,2,9,11] or [12] for details).

The paper is organized as follows. After introducing several necessary algebraic concepts in Section 2, we introduce in Section 3 fuzzy dynamic operators in an arbitrary fuzzy logic whose semantics is axiomatized by means of a residuated poset. In Section 4 we get a simple construction of fuzzy dynamic operators which uses lattice theoretical properties of the underlying ordered set. In the case when the underlying residuated poset is not a complete lattice, we show how to apply the lattice completion to this construction. In Sections 5 and 6 we solve the problem of an approximation and a representation of fuzzy dynamic algebras. This means that we get a procedure how to construct a corresponding time frame to be in accordance with the construction from Section 4. In particular, any fuzzy dynamic algebra is representable in its Dedekind–MacNeille completion. Section 7 is devoted to a discussion about approx-