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Pavelka-style fuzzy logic in retrospect and prospect

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Dedicated to Lotfi Zadeh

Abstract

We trace the origin and development of Pavelka-style fuzzy logic, discuss its significance and clarify some related misconceptions.

Keywords: Fuzzy logic; Pavelka-style logic; Vagueness; Uncertainty

1. Introduction

Shortly after Zadeh's seminal 1965 paper on fuzzy sets [41], two papers by Joseph Goguen appeared. Goguen's remarkable, visionary publications—the *L-fuzzy sets* of 1967 [19] and *The logic of inexact concepts* of 1968 [20]—led to a number of developments of various topics in fuzzy logic. Among the most important ones is the development of fuzzy logic as a logical calculus to which Goguen's papers provided an original, inspiring conception but also specific technical contributions. Goguen's conception served as an inspiration to Jan Pavelka who developed it in a series of papers [29–31] in the late 1970s. The resulting *Pavelka-style (fuzzy) logic* nowadays represents an important branch of fuzzy logic in the narrow sense. The 50th anniversary of the publication of Zadeh's seminal paper, which is considered the birth of fuzzy logic, provides an opportunity to reflect upon the origin, the development as well as the still prevailing misconceptions regarding Pavelka-style logic. A brief account of these is the subject of this paper.

2. Roots of Pavelka-style logic

2.1. Two sources

Pavelka-style logic has its roots in the two aforementioned Goguen's papers, particularly [20], and in Tarski's view of a deductive system as a consequence operator [35]. Thanks to the latter, which allows a very general view of a

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logical calculus,[1] Pavelka-style logic represents a considerably broad logical framework subsuming diverse formal systems. Thanks to the former, to which I now turn, Pavelka-style logic naturally embodies the idea of approximate reasoning, in particular of what became known as a graded approach to truth.

2.2. *Goguen's seminal contribution*

Ever since their appearance, Goguen's papers [19,20] have been and continue to be—almost fifty years after their publication—an essential reading in fuzzy logic. This is so due to their clarity, a clean treatment of a broad spectrum of conceptual issues, as well as a reasonable technical depth. Here I restrict to the aspects directly related to Pavelka-style logic and refer to the upcoming book [2] for more information.

When regarded from a logical and mathematical viewpoint, Goguen's work represents a natural development of the initial ideas regarding fuzzy sets by Zadeh, of whom Goguen was a doctoral student at Berkeley. Its logical part differs substantially from the then recent contributions to many-valued logic in an important aspect. Namely, most of these contributions did not pay attention to the meaning of additional truth values. This was not true of some of the original contributions to many-valued logic, namely those by Łukasiewicz, Kleene, Bochvar and others who were guided by a more or less clear meaning of truth values employed in their logics, see [21,25] and [2]. However, in the other and later contributions to many-valued logic, interpretation of truth values has often been considered problematic or not not considered at all. Consequently, most of these contributions were in fact developing systems of many-valued logic which were, however mathematically sophisticated, somewhat sterile. Goguen, on the other hand, consistently interpreted truth values as degrees of truth and, in a sense, derived the novel conception of his "logic of inexact concepts" from this interpretation and from concrete problems in human reasoning. I consider this a remarkable and noteworthy moment in the development of many-valued logic and fuzzy logic in particular demonstrating a commonly encountered situation in mathematics and logic—a reasonable practical motivation guiding in a search for and eventually leading to a conceptually new and useful theory.

One concrete problem in human reasoning Goguen considered was the well-known sorites paradox, for which he refers to Max Black's treatment [7]. Black, who significantly contributed to the understanding of vagueness by his seminal contribution [6] in which he proposed predecessors of fuzzy sets, so called consistency profiles, may in this respect be considered an important source of inspiration for Pavelka-style fuzzy logic.

Leaving details aside, the crux of Goguen's resolution of the sorites paradox derives from the following observation [20, pp. 336, 371]:

> Just as propositions . . . are no longer either 'true' or 'false', but can be intermediate, deductions are no longer 'valid' or 'invalid'. . . . It seems to us that the present methods provide a framework for an 'inexact mathematics' in which we can apply approximately valid deductive procedures to approximately true hypotheses. The 'logic of inexact concepts' then helps assess the validity of the final conclusions.

In particular, Goguen examines in presence of a scale L of truth degrees the deduction rule of *modus ponens*, whose classical form reads "from φ and $\varphi \to \psi$ infer ψ" and is displayed as $\frac{\varphi, \varphi \to \psi}{\psi}$[2]:

> First, how is deduction used? We have a truth value $[P]$ and a truth value $[P \Rightarrow Q]$, and we want to estimate the truth value $[Q]$;

arriving thus at a more general rule

$$\frac{\varphi \text{ with degree } a, \varphi \to \psi \text{ with degree } c}{\psi \text{ with degree } a \otimes^{\cdot} c} \tag{1}$$

where $\otimes^{\cdot}$ is a truth function of a many-valued conjunction $\otimes$ adjoint to $\to$. One may observe that in the classical case, i.e. $L = \{0, 1\}$, Goguen's *modus ponens* may be identified with the ordinary one. The rule also has the intuitively appealing property that the result of "a long chain of only slightly unreliable deductions can be very un-

[1] See e.g. [39].

[2] He denotes formulas by P, Q, etc., implication by $\Rightarrow$, and the truth value of P by $[P]$. We denote formulas by $\varphi, \psi, \ldots$, logical connectives by $\to$ (implication), $\otimes$ (conjunction), . . . , their truth functions by $\to^{\cdot}$, $\otimes^{\cdot}$, . . . , and the truth value of formula φ by $\|\varphi\|$.