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# A unifying framework for duality and modeling in robust linear programs

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ABSTRACT

In this paper, our major theme is a unifying framework for duality in robust linear programming. We show that there are two pair of dual programs allied with a robust linear program; one in which the primal is constructed to be "ultra-conservative" and one in which the primal is constructed to be "ultra-optimistic." Furthermore, as one would expect, if the uncertainly in the primal is row-based, the corresponding uncertainty in the dual is column-based, and vice-versa. Several examples are provided that illustrate the properties of these primal and dual models.

A second theme of the paper is about modeling in robust linear programming. We replace the ordinary activity vectors (points) and right-hand sides with well-known geometric objects such as hyper-rectangles, parallel line segments and hyper-spheres. In this manner, imprecision and uncertainty can be explicitly modeled as an inherent characteristic of the model. This is in contrast to the usual approach of using vectors to model activities and/or constraints and then, subsequently, imposing some further constraints in the model to accommodate imprecision and uncertainties. The unifying duality structure is then applied to these models to understand and interpret the marginal prices. The key observation is that the optimal solutions to these dual problems are comprised of two parts: a traditional "centrality" component along with a "robustness" component.



## 1. Introduction

Throughout the history of linear programming, duality and the resulting marginal prices have been a fundamental tool of linear programming theory and practice. In the emerging field of robust linear programming, the role of duality has not been central to the development of the theory. One explanation for this observation may stem from the fact that applications of the methodologies have not yet been used in a large number of applications. Hence, both the need and opportunities to explain the sensitivity of optimal solutions have been limited. However, certain elements of duality for robust linear programs have existed for decades and some new results have recently been developed.

Nearly 50 years ago, the notion of replacing the ordinary points (vectors) $a_j$, $b \in R^m$ in a linear program

$$\begin{aligned} \max \quad & c_1x_1 + \cdots + c_nx_n \\ & x_1a_1 + \cdots + x_na_n \le b \\ & x_j \ge 0 \end{aligned} \tag{1}$$

with sets was introduced by Dantzig [8] in the form of what was termed a "generalized linear program GLP." A GLP is obtained from (1) by allowing the decision-maker to "freely choose" any activity vector $a_j$ from a convex set $K_j$. In other words, the decision-maker has the freedom to choose the best outcome $a_j \in K_j$, $j=1,2,\ldots,n$, along with the decision variables $x_j$.

A decade later, Soyster [12] considered (1) when the decision-maker must accommodate all possible outcomes $a_j \in K_j$. Soyster termed this problem "inexact linear programming" where the ordinary algebraic inequalities were replaced by a set-inclusive formulation (2), namely,

$$\begin{aligned} \max \quad & c \cdot x \\ & x_1K_1 \oplus x_2K_2 \oplus \cdots \oplus x_nK_n \subseteq K \\ & x_j \ge 0 \end{aligned} \tag{2}$$

where the sets $\{K_j\}$ and $K$ are closed convex sets and the operator $\oplus$ refers to the addition of sets, i.e.,

$$K_1 \oplus K_2 = \{x_1 + x_2 \,|\, x_1 \in K_1,\ x_2 \in K_2\}$$

and for $a \ge 0$,

$$aK_1 = \{ax_1 \,|\, x_1 \in K_1\}.$$

This set operator has two properties that we use: $K_1 \oplus K_2$ is a convex set and $(a+b)K = aK \oplus bK$; $a,b \ge 0$. The sets $\{K_j\}$ are possible values for the activity coefficients and the set $K$ includes all acceptable values for the right-hand side. A feasible solution $\bar{x}$ to (2) means that $A\bar{x} \in K$ for all possible realizations $A=(a_1, a_2,\ldots,a_n)$ of the constraint matrix.

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While GLP is a best case (optimistic) model, since the decision-maker can choose $a_j$ along with $x_j$, the inexact linear programming model (2) is just the opposite, a worst-case, conservative approach. The meaning and contrast between these two problems, GLP and inexact linear programs, are the motivation and basis for this paper. In particular, what is the interpretation of the comparison between the "duals" of these two problems? Furthermore, how do any such insights relate to contemporary results in linear robust optimization?

For the special case in which

$$K = K\left(\bar{b}\right) = \{b \in R^m \,|\, b \le \bar{b}\},$$

the set inclusive problem (2) reduces to an ordinary linear program of the form

$$\begin{aligned} \max \quad & c \cdot x \\ & \bar{A}x \le \bar{b} \\ & x \ge 0 \end{aligned} \tag{3}$$

for a specifically constructed matrix $\bar{A}$. (The $j$th column of the matrix $\bar{A}$ is generated by finding the least upper bound of each component $a_{ij}$ for $a_j \in K_j$. See Soyster [12].) Program (2) has been characterized as the "column-wise" approach to uncertainty (for obvious reasons).

Interest in this class of problems was dormant until the mid-1990s and the publication of several seminal papers, particularly Ben-Tal and Nemirovski [2,3]. Since that time, the field, now known as "robust optimization", has grown enormously and comprehensive reviews of its development and extension are chronicled in [4–6].

Ben-Tal and Nemirovski [3] introduce the notion of "row-wise" uncertainty, namely,

$$\begin{aligned} \min \quad & u \cdot b \\ & u \cdot a_j \ge c_j \quad \forall a_j \in K_j \end{aligned} \tag{4}$$

Under certain conditions, Ben-Tal and Nemirovski [3] show that (4) is infeasible if and only if there is some realization $\{\bar{a}_1, \ldots, \bar{a}_n\}$ which renders the corresponding linear program infeasible. The "row-wise" program (4), as originally presented, has become the de facto definition of linear robust optimization, i.e., it has been accepted that "row-wise" uncertainty is the more general case. Nevertheless, both column and row uncertainty are important to model. Row-wise uncertainty captures the uncertainty in the transformation coefficients from an input or to an output. Column-wise uncertainty represents uncertainty in processes, the activity-analysis view of Dantzig [8].

The development and contrast between row and column uncertainty models lead directly to the major focus of this paper, a duality framework for linear robust optimization for both the "column" (2) and "row" problems (4). Although there are some earlier results for duality in robust linear programming, [9,13–16], a unified understanding seems to be missing. However, the recent paper by Beck and Ben-Tal [1] provides a key insight which is discussed in the next section.

Minoux [11] offers an example which seems to discount the usefulness of duality in this robust linear programming context. Consider the following problem from Minoux [11]:

$$\begin{aligned} \max \quad & 4x_1 + 3x_2 \\ & a_1x_1 + a_2x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{5}$$

where $a_1 \in [2,3]$ and $a_2 \in [1,2]$. To ensure feasibility for any realization of $a_1 \in [2,3]$, $a_2 \in [1,2]$, one chooses $x_1^* = 0$, $x_2^* = 2$ with optimal objective value 6. Next, consider a proposed dual from Minoux [11]:

$$\begin{aligned} \min \quad & 4u \\ & ua_1 \ge 4 \\ & ua_2 \ge 3 \\ & u \ge 0 \end{aligned} \tag{6}$$

and to ensure feasibility, choose $u^* = 3$, so the optimal objective value is 12. There is a gap.

Observe that both the primal (5) and dual (6) are formulated to ensure feasibility; in essence, both formulations are "conservative." On the other hand, note that if one proposes a dual for (5) for the most optimistic outcome of the uncertain matrix, one would choose $a_1 = 3$, $a_2 = 2$, i.e.

$$\begin{aligned} \min \quad & 4u \\ & 3u \ge 4 \\ & 2u \ge 3 \\ & u \ge 0 \end{aligned} \tag{6a}$$

for which $u^* = 3/2$ is optimal with objective value 6. Beck and Ben-Tal [1] address this gap with the introduction of an "optimistic feasible solution." This is a solution which is feasible for some realization of the uncertainly set. In this case the realization in (6) is $(a_1, a_2) = (3,2)$ and the "optimistic feasible solution" is $u = 3/2$. Then, the "optimistic counterpart" is the problem of finding the best solution from among all "optimistic solutions," which in this case is the solution obtained from (6a). Note that in this case, the result is that the value of the "worst case" for the primal equals the value of the "best case" for the dual, the major theme in Beck and Ben-Tal [1]. However, Beck and Ben-Tal [1] show that the "optimistic counterpart" is not always well-behaved in that the formulation may not result in a convex program. Consider the simple one-constraint robust linear program

$$\begin{aligned} \min \quad & c \cdot x \\ & a_1x_1 + a_2x_2 \ge 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

where $(a_1, a_2)$ are points on the line segment connecting the points (1,0) and (0, 1). Note that $\bar{x}_1 = (1,0)$ is an optimistic feasible solution (allied with outcome $a = (1,0)$) and $\bar{x}_2 = (0,1)$ is an optimisitic solution (allied with the outcome $a = (0,1)$). But no outcome makes $(\bar{x}_1 + \bar{x}_2)/2$ optimistically feasible. It is interesting to note that this non-convexity result is for a "row" problem, i.e. the uncertainty set is row-based rather than column based. The source of the non-convexity can be seen through reformulating the problem as a standard optimization with $a_1$, $a_2$ becoming variables

$$\begin{aligned} \min \quad & c \cdot x \\ & a_1x_1 + a_2x_2 \ge 1 \\ & a_1 + a_2 = 1 \\ & a_1, a_2 \ge 0 \\ & x_1, x_2 \ge 0 \end{aligned}$$

The original constraint has two bilinear terms.

This dilemma from Minoux [11] and resolution by Beck and Ben-Tal [1] motivate a major theme of this paper. Given a linear program with uncertain data, there are always two (extreme) formulations—one we denote as "ultra-conservative" and the other "ultra-optimistic" (where the use of the term "ultra" refers to the extreme bound). The ultra-optimistic formulation is essentially a GLP and the ultra-conservative formulation is the set-inclusive formulation (2). In Section 2, these two formulations are addressed in detail with a focus on their respective dual programs where the ultra-optimistic and ultra-conservative characterizations are reversed. Section 3 addresses how the duality results of Section 2 apply to a