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Decreasing downside risk aversion and background risk

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HIGHLIGHTS

- Risk vulnerability amounts to reducing risk taking in the presence of background risk.
- We associate risk vulnerability to decreasing downside risk aversion (DDRA).
- We show that DDRA in the Arrow–Pratt sense is necessary to obtain risk vulnerability.
- We also demonstrate that Ross-DDRA is sufficient to have risk vulnerable preferences.

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ABSTRACT

In this paper, we show that risk vulnerability can be associated with the concept of downside risk aversion (DRA) and an assumption about its behavior, namely that it is decreasing in wealth. Specifically, decreasing downside risk aversion in the Arrow–Pratt and Ross senses are respectively necessary and sufficient for a zero-mean background risk to raise the aversion to other independent risks.

0. Introduction

Doherty and Schlesinger (1983) were the first to address the impact of zero mean background risks on the demand for insurance. Since then, the way that background risks modify the propensity to make risky decisions has attracted a great deal of attention. Among other papers on this topic,[1] Eeckhoudt and Kimball (1992), Gollier and Pratt (1996) and Eeckhoudt et al. (1996) establish necessary and/or sufficient conditions on the shape of the utility function so that the behavior seen as the most plausible (i.e. background risks raise risk aversion, a preference termed "risk vulnerability") is obtained. There is a variety of assumptions made about background risks in these papers. For instance, Eeckhoudt and Kimball (1992) analyze the effect of the introduction of a zero-mean background risk possibly correlated to the foreground one. Gollier and Pratt (1996) consider the addition of fair or unfair background risks while Eeckhoudt et al. (1996) examine the impact of first- and second-order dominance shifts in background risk. This literature about background risks was developed mainly in the 1990s and it was summarized by Gollier (2001) (see especially chapters 8 and 9).

More recently, a measure of the intensity of absolute downside risk aversion (DRA) was introduced by Modica and Scarsini (2005) and gave rise to various developments, including its extension to higher orders.[2] In this paper, we indicate that risk vulnerability can be associated with the concept of downside risk aversion. Specifically, to obtain that risk taking falls when the decision maker faces an independent zero mean background risk: (1) decreasing downside risk aversion (DDRA) in the Arrow (1965) and Pratt

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[1] See for example Pratt and Zeckhauser (1987) and Kimball (1993) who define the related concepts respectively of properness and of standardness.

[2] See for instance Jindapon and Neilson (2007), Crainich and Eeckhoudt (2008), Li (2009), Denuit and Eeckhoudt (2010), Wang and Li (2010) and Liu and Meyer (2013) for the introduction, the interpretation and the use of this concept.

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(1964) sense is necessary and; (2) DDRA in the Ross (1981) sense is sufficient.

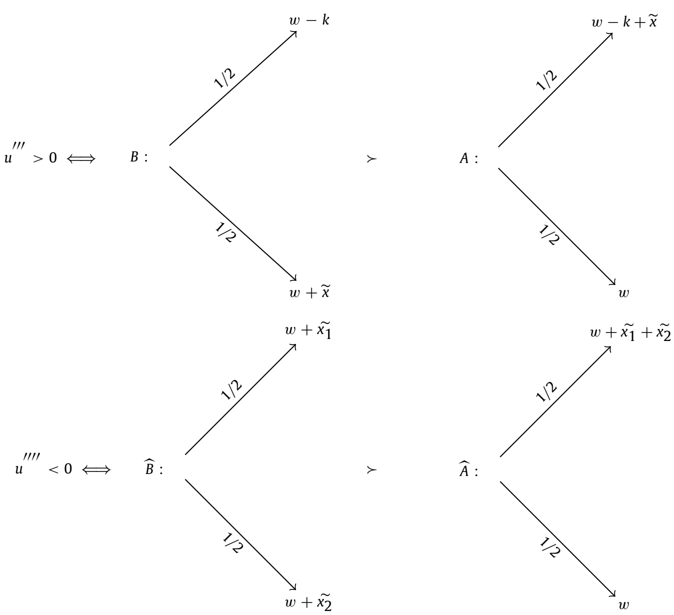
Our contribution thus relates risk vulnerability to an easily interpretable concept. As recently pointed out by Huang (2012) the concept of DRA is theoretically appealing but it has yet to be associated with a specific economic behavior. We show in this paper that in fact downside risk aversion and the measurement of its intensity are appropriate concepts to analyze the impact of background risks.

The paper is organized as follows. In Section 1, we define the concepts that are used in the proofs, essentially downside risk aversion and temperance. Then, we formalize the assumption of decreasing downside risk aversion. In Section 2, we present the structure of the problem and after some preliminary results (Section 2.1), we prove in Section 2.2 the central result of the paper. We conclude in Section 3.

1. Downside risk aversion and temperance

The concepts of downside risk aversion or, equivalently, prudence ($u''' > 0$[3]), and temperance ($u'''' < 0$) were introduced respectively by Menezes et al. (1980) and by Kimball (1992).

These two concepts can be interpreted as a preference for the disaggregation of pains (Eeckhoudt and Schlesinger, 2006). For downside risk aversion, the pains are either a sure loss ($-k$) or a zero mean risk ($\widetilde{x}$) while for temperance the pains are two independent zero mean risks ($\widetilde{x}_1$ and $\widetilde{x}_2$). More precisely we have under expected utility (EU)



Observe that in the lotteries B and $\widehat{B}$ the pains never occur jointly (they are disaggregated), which is not the case for lotteries A and $\widehat{A}$ where they appear together in the same state of the world.

While the signs of these derivatives indicate a "direction" for preference, a measure of their intensities is needed to make them operational.[4]

For downside risk aversion, the measure of intensity[5] is $\frac{u'''}{u'}$.[6] To derive this measure, we start with the lotteries A and B described above and express the difference in their expected utility by R where

$$R = \frac{1}{2}\left[u(w-k) + E[u(w+\widetilde{x})]\right] - \frac{1}{2}\left[E[u(w-k+\widetilde{x})] + u(w)\right] \gtreqless 0 \quad \text{if } u''' \gtreqless 0. \tag{1}$$

Expanding to second order terms that involve $\widetilde{x}$, we have

$$R \simeq \frac{1}{2}\frac{\sigma_{\widetilde{x}}^2}{2}[u''(w) - u''(w-k)], \tag{2}$$

or for small k we have

$$R \simeq \frac{1}{4}\sigma_{\widetilde{x}}^2 k u'''(w). \tag{3}$$

Returning to Eq. (1), R can be made to equal zero under $u''' > 0$ by adding a positive amount of money m to w in $u(w)$. The m is thus the amount of money necessary to compensate the downside risk averse individual for the risk misapportionment. Using the technique presented by Crainich and Eeckhoudt (2008), we obtain

$$m \simeq \frac{1}{4}\sigma_{\widetilde{x}}^2 k \frac{u'''(w)}{u'(w)} \tag{4}$$

where $\frac{u'''(w)}{u'(w)}$ is the coefficient of downside risk aversion.

The assumption of decreasing DRA (DDRA in short) implies

$$\forall w \quad \frac{\partial}{\partial w}\left(\frac{u'''(w)}{u'(w)}\right) \leq 0 \tag{5}$$

that is equivalent to

$$-\frac{u''''(w)}{u'''(w)} \geq -\frac{u''(w)}{u'(w)}. \tag{6}$$

The left-hand side of (6) is the coefficient of absolute temperance (denoted T) while the right-hand side is the well-known coefficient of absolute risk aversion (denoted A).

All of the measures of intensity presented so far are developed in a framework similar to the one adopted by Arrow (1965) and Pratt (1964) in their seminal papers about risk aversion. However, an alternative approach is suggested by Ross (1981) who proposes a stronger measure of absolute risk aversion. The developments we make in the following sections refer to these stronger measures not only for risk aversion but also for downside risk aversion and temperance.

A detailed analysis of these higher order extensions of Ross' contribution (1981) is beyond the scope of this paper.[7] For our purpose, u is defined as being more downside risk averse than v in the Ross' sense if

$$\forall t\ \forall w \quad \frac{u'''(w+t)}{v'''(w+t)} \geq \frac{u'(w)}{v'(w)}. \tag{7}$$

Then decreasing downside risk aversion in the sense of Ross (Ross DDRA) corresponds to temperance being at least equal to risk aversion in the Ross' sense[8] that is:

$$\forall t\ \forall w \quad -\frac{u''''(w+t)}{u'''(w+t)} \geq -\frac{u''(w)}{u'(w)}. \tag{8}$$

i.e. absolute temperance at $w+t$ exceeds risk aversion at w for all w and t.[9]

[3] u''' and u'''' represent respectively the third and fourth derivatives of the utility function u. u' and u'' denote the first and second derivatives.

[4] Compare with risk aversion defined by $u'' < 0$ while its intensity is given by the concept of absolute risk aversion.

[5] In the rest of the paper, the intensity of downside risk aversion is denoted by DRA.

[6] Compare with the intensity of prudence $-\frac{u'''}{u''}$ proposed by Kimball (1990).

[7] The interested reader can refer to the elegant and useful synthesis given in Section 3 of Liu and Meyer (2013).

[8] This statement parallels the Ross' definition of decreasing absolute risk aversion which corresponds to prudence exceeding risk aversion in the Ross' sense (see Theorem 4 in Ross (1981)).

[9] Liu–Meyer's paper refers in general to (n/m)th degree Ross more risk averse order. In our case $n = 4$ and $m = 2$.