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Optimal allocation of policy limits and deductibles

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Abstract

In this paper, we study the problems of optimal allocation of policy limits and deductibles. Several objective functions are considered: maximizing the expected utility of wealth assuming the losses are independent, minimizing the expected total retained loss and maximizing the expected utility of wealth when the dependence structure is unknown. Orderings of the optimal allocations are obtained.

Keywords: Arrangement increasing functions; Stochastic order; Dependence structure; Comonotonicity

1. Introduction

Suppose that a policyholder is exposed to n random losses (risks). Through paying a premium, (s)he could obtain coverage from an insurer. Two common forms of coverage are (ordinary) deductible and policy limit (cf. Klugman et al. (2004)). In some situations, the policyholder has the right to allocate the deductibles or the policy limits among the n risks. For example, the compensation package of many big companies includes a commonly called "Flexible Spending Account Programme", which allows employees to allocate pre-tax dollars toward specific expenses such as healthcare, medical costs or dependent care. This is essentially a form of allocating policy limits. This paper addresses the problem of finding the optimal way to allocate policy limits and deductibles. While closed-form expression of the optimal allocation is out of reach in general because of the nonlinear nature of the problem, we could still obtain useful qualitative results concerning the relative size of each allocation.

The paper is organized as follows. In Section 2, we recall some basic results about arrangement increasing functions and stochastic dominance orders. Section 3 studies the problem of optimal allocation of policy limits. Section 4 gives a parallel treatment for deductibles. Section 5 concludes the paper.

2. Preliminary

In this section, we will collect some facts concerning arrangement increasing functions, stochastic dominance orders and comonotonicity that are useful in the sequel.

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2.1. Arrangement increasing functions

Let τ be any permutation of the set $\{1, 2, \ldots, n\}$. For any vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we use $\mathbf{x} \circ \tau$ to denote the permuted vector $(x_{\tau(1)}, \ldots, x_{\tau(n)})$. Let $x_{[1]} \geq \cdots \geq x_{[n]}$ denote the components of $\mathbf{x}$ in decreasing order and let $\mathbf{x}_\downarrow = (x_{[1]}, \ldots, x_{[n]})$ denote the decreasing rearrangement of $\mathbf{x}$. Similarly, let $x_{(1)} \leq \cdots \leq x_{(n)}$ denote the components of $\mathbf{x}$ in increasing order and let $\mathbf{x}_\uparrow = (x_{(1)}, \ldots, x_{(n)})$ denote the increasing rearrangement of $\mathbf{x}$.

Definition 1. A real-valued function $g(\mathbf{x}, \boldsymbol{\lambda})$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be an *arrangement increasing (AI) function* if

(1) g is permutation invariant, i.e. $g(\mathbf{x}, \boldsymbol{\lambda}) = g(\mathbf{x} \circ \tau, \boldsymbol{\lambda} \circ \tau)$ for any permutation τ, and
(2) g exhibits permutation order, i.e. $g(\mathbf{x}_\downarrow, \boldsymbol{\lambda}_\uparrow) \leq g(\mathbf{x}_\downarrow, \boldsymbol{\lambda} \circ \boldsymbol{\tau}) \leq g(\mathbf{x}_\downarrow, \boldsymbol{\lambda}_\downarrow)$ for any permutation τ.

Property (2) means that an AI function attains its maximum value when $\mathbf{x}$ and $\boldsymbol{\lambda}$ are similarly ordered, and it attains its minimum value when they are oppositely ordered. The most prominent example of AI function is

$$g(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} x_i \lambda_i .$$

The fact that this is an AI function is indeed the well-known rearrangement inequality.

A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called supermodular (or L-superadditive) if $\phi(r + \eta, s) - \phi(r, s)$ is increasing in s for all r and all $\eta > 0$. The next two results are due to Hollander et al. (1977).

Lemma 1. *A function* $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *having the form*

$$g(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} \phi(x_i, \lambda_i)$$

is AI if ϕ *is supermodular.*

Lemma 2. *A function* $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *having the form*

$$g(\mathbf{x}, \boldsymbol{\lambda}) = \psi(\mathbf{x} - \boldsymbol{\lambda})$$

is AI if $\psi : \mathbb{R}^n \to \mathbb{R}$ *is Schur-concave.*

For a thorough and detailed account of the above concepts, we refer to Marshall and Olkin (1979).

2.2. Stochastic dominance orders

Standard references for this subsection include Denuit et al. (2005), Kaas et al. (1994, 2001), Müller and Stoyan (2002), and Shaked and Shanthikumar (1994). Throughout this paper, all the random variables considered are defined on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. We also assume that all the expectations mentioned exist. For any random variable X, F_X denotes its distribution function: $F_X(t) = \mathbb{P}(X \leq t)$.

Definition 2. Let X and Y be two random variables.

1. Y is said to be larger than X in the *usual stochastic order* (resp. *increasing convex order, convex order*), denoted as $Y \geq_{st} X$ (resp. $Y \geq_{icx} X$, $Y \geq_{cx} X$), if

$$\mathbb{E}[f(Y)] \geq \mathbb{E}[f(X)]$$

for all increasing (resp. increasing convex, convex) function f.

2. Y is said to be larger than X in the *hazard rate order*, denoted as $Y \geq_{hr} X$, if

$$\frac{1 - F_Y(s)}{1 - F_X(s)} \quad \text{is nondecreasing in } s.$$

The following lemma gives a useful characterization of the usual stochastic order.