Insurance: Mathematics and Economics 47 (2010) 167–175

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

On optimal allocation of risk vectors

Swen Kiesel, Ludger Rüschendorf*

Department of Mathematical Stochastics, University of Freiburg, Eckerstraße 1, 79104 Freiburg, Germany

ARTICLE INFO

Article history:
Received March 2010
Received in revised form
May 2010
Accepted 27 May 2010

JEL classification:
G32
C43

MSC:
91B30
90C46

Keywords:
Optimal risk allocations
Worst case portfolio
Comonotonicity

ABSTRACT

In this paper we extend results on optimal risk allocations for portfolios of real risks w.r.t. convex risk functionals to portfolios of risk vectors. In particular we characterize optimal allocations minimizing the total risk as well as Pareto optimal allocations. Optimal risk allocations are shown to exhibit a worst case dependence structure w.r.t. some specific max-correlation risk measure and they are comonotone w.r.t. a common worst case scenario measure. We also derive a new existence criterion for optimal risk allocations and discuss some examples.

1. Introduction

In this paper we consider an extension of the optimal risk allocation problem resp. the risk exchange problem to the case of risk vectors. This extension allows one to include the effects of dependence in a portfolio as measured by multivariate risk measures ϱ_i in the risk allocation problem. On a basic probability space $(\Omega, \mathcal{A}, P)$ we consider convex, proper, normed lower semicontinuous (lsc) risk functions, called in the following *risk functionals* $\varrho_i : L_d^p(P) \to (-\infty, \infty]$, $1 \le i \le n$, defined on risk vectors $X = (X^1, \dots, X^d)$ with $X^i \in L^p(P) = L^p$, i.e. $L_d^p(P) = L_d^p$ is the d-fold product of $L^p(P)$. The risk functionals ϱ_i describe the risk evaluation of n traders in the market. Here ϱ_i are *normed* means that $\varrho_i(0) = 0$ and ϱ_i are *proper* means that $\operatorname{dom} \varrho_i \neq \emptyset$ and $\varrho_i(X) \neq -\infty$ for all X. We allow unbounded risks and assume that $1 \le p \le \infty$.

For a given portfolio of d risks described by a risk vector $X = (X^1, \dots, X^d) \in L_d^p$ we define the set $\mathcal{A}(X) = \mathcal{A}^n(X)$ of *n-allocations* of the portfolio X by

$$\mathcal{A}(X) := \left\{ (\xi_1, \dots, \xi_n) \mid \xi_i \in L_d^p, \sum_{i=1}^n \xi_i = X \right\}. \tag{1.1}$$

For an allocation $(\xi_1, \dots, \xi_n) \in \mathcal{A}(X)$ trader i is exposed to the risk vector ξ_i which is evaluated by the risk functional ϱ_i. ξ_i may contain some zero components and thus trader i may only be exposed to some of the d components of risk in our formulation. Let

$$\mathcal{R} := \{(\varrho_i(\xi_i)) \mid (\xi_i) \in \mathcal{A}(X)\} \tag{1.2}$$

denote the corresponding risk set. Our aim is to characterize Pareto-optimal (PO) allocations $(\xi_i) \in \mathcal{A}(X)$, i.e. allocations such that the corresponding risk vectors are minimal elements of the risk set $\mathcal{R}$ in the pointwise ordering. A related optimization problem is to characterize allocations $(\eta_i) \in \mathcal{A}(X)$ which minimize the total risk, i.e.

$$\sum_{i=1}^n \varrho_i(\eta_i) = \inf \left\{ \sum_{i=1}^n \varrho_i(\xi_i) \mid (\xi_i) \in \mathcal{A}(X) \right\} =: \bigwedge \varrho_i(X). \tag{1.3}$$

The optimal allocation problem of risks is a classical problem in mathematical economics and insurance and is of considerable practical and theoretical interest. It has been studied in the case of real risks, i.e. in the case $d = 1$ in the classical papers of Borch (1962), Gerber (1979), Bühlmann and Jewell (1979), Deprez and Gerber (1985) and others in the context of risk sharing in insurance and reinsurance contracts. In more recent years this problem has also been studied in the context of financial

* Corresponding author. Tel.: +49 0 761 203 5665; fax: +49 0 761 203 5661.
E-mail address: ruschen@stochastik.uni-freiburg.de (L. Rüschendorf).

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doi:10.1016/j.insmatheco.2010.05.005

risks as in risk exchange, assignment of liabilities to daughter companies, individual hedging problems and others (see the papers of Heath and Ku (2004), Barrieu and El Karoui (2005), Burgert and Rüschendorf (2006, 2008), Jouini et al. (2007), Acciaio (2007), Filipović and Svindland (2008), KR[1] (2008), and others).

The aim of this paper is to extend the risk allocation results to the case of multivariate risks resp. the case of risk portfolios. The main motivation for considering multivariate risk measures is to include the influence of (positive) dependence on the risk of a portfolio. In recent papers several of aspects of multivariate risks such as worst case portfolios, diversification effects or strong coherence have been studied (see e.g. Ekeland et al. (2009), R (in press), Carlier et al. (2009)). As we will see the optimal risk allocation problem has some close ties to these developments.

After the introduction of some basic notions from convex analysis in Section 2 we derive in Section 3 the basic characterization of optimal total risk minimizing allocations and give a link to Pareto-optimal allocations. Due to the multivariate structure the proof of this characterization needs a new element in the analysis. In Section 4 we specialize to law invariant convex risk measures ϱ_i. A characterization of their subgradients leads to a close connection between optimal allocations and worst case portfolio vectors. More precisely it is shown that optimal allocations are comonotone w.r.t. a common worst case scenario measure. Further they exhibit a worst case dependence structure w.r.t. some specific max-correlation risk measure. In Section 5 we derive a new general existence criterion for optimal allocations and finally discuss some examples in Section 6.

2. Some notions from convex analysis

Throughout this paper we consider convex, lower semicontinuous (lsc) proper risk functions ϱ on L^p_d, called in the following risk functionals. Generally, for a convex proper function on a locally convex space E paired with its dual space E^* by $(E, E^*, \langle\cdot,\cdot\rangle)$ we denote by

$$f^* : E^* \to \overline{\mathbb{R}}, \qquad f^*(x^*) = \sup_{x\in E}(\langle x^*, x\rangle - f(x)), \quad x^* \in E^*, \tag{2.1}$$

the convex conjugate and by

$$f^{**} : E \to \overline{\mathbb{R}}, \qquad f^{**}(x^*) = \sup_{x^*\in E^*}(\langle x, x^*\rangle - f^*(x^*)), \quad x \in E, \tag{2.2}$$

the bi-conjugate of f. Let further

$$\partial f(x) = \{x^* \in E^* \mid f(y) - f(x) \geq \langle x^*, y - x\rangle, \forall y \in E\} \tag{2.3}$$

denote the set of subgradients of f in x. Then

$$\partial f(x) = \{x^* \in E^* \mid \forall y \in E, \langle x^*, y\rangle \leq D(f; x)(y)\}, \tag{2.4}$$

where $D(f, x)(y)$ is the right directional derivative of f in x in direction y. This connection is useful in the applications in order to calculate the subgradient.

For a proper convex function f with $\partial f(x) \neq \emptyset$ it holds that

$$x^* \in \partial f(x) \Leftrightarrow \langle x^*, x\rangle = f^*(x^*) + f(x) \tag{2.5}$$

$$\Leftrightarrow \langle x^*, x\rangle - f(x) = \sup_{y\in E}(\langle x^*, y\rangle - f(y)). \tag{2.6}$$

Thus x is a minimizer of f if and only if

$$0 \in \partial f(x) \quad \text{(Fermat's rule)}. \tag{2.7}$$

[1] Kiesel and Rüschendorf is abbreviated within this paper with [KR], Rüschendorf with [R].

If f is furthermore lower-semicontinuous, then we get by the Fenchel–Moreau-Theorem the equivalence:

$$0 \in \partial f(x) \Leftrightarrow x \in \partial f^*(0). \tag{2.8}$$

$\partial f^*(0)$ is the set of all minimizers of f.

For convex, lsc proper risk functionals ϱ_i it follows that ϱ_i^* are proper and

$$\left(\bigwedge \varrho_i\right)^* = \sum_{i=1}^n \varrho_i^* \tag{2.9}$$

(see Barbu and Precupanu (1986, Section 2)). In consequence we obtain

$$\bigcap_{i=1}^n \operatorname{dom} \varrho_i^* \neq \emptyset \Rightarrow \operatorname{dom}(\wedge\varrho_i) \neq \emptyset \tag{2.10}$$

(see KR (2008, Proposition 2.1)). For all results on convex duality we refer to Rockafellar (1974) and Barbu and Precupanu (1986).

In this paper we deal with the dual pair $(L^p_d, L^q_d, \langle\cdot,\cdot\rangle_d)$ where $\langle\cdot,\cdot\rangle_d$ denotes the canonical scalar product on the product spaces

$$\langle Z, X\rangle_d := \sum_{j=1}^d EZ^jX^j \tag{2.11}$$

for $X = (X^1, \dots, X^d) \in L^p_d, Y = (Y^1, \dots, Y^d) \in L^q_d$, where q is the conjugate index to p, $\frac{1}{p} + \frac{1}{q} = 1$. In the case $p = 1, q = \infty$ the dual space is the set ba^q_d of d-tuples of finitely additive normal P-continuous measures integrating $|x|^q$. To avoid cumbersome notation we still use L^q_d in this case. For law invariant risk functionals, as used in the second part of the paper, in fact we can reduce to the class of probability measures and thus to L^q_d.

The portfolio vectors $(\xi_i)_{1\leq i\leq n}$ are contained in the corresponding productspaces defining the dual pair $((L^p_d)^n, (L^q_d)^n, \langle\cdot,\cdot\rangle^n_d)$ where for $X = (X_1, \dots, X_n) \in (L^p_d)^n, Z = (Z_1, \dots, Z_n) \in (L^q_d)^n$ the scalar product is given by

$$\langle Z, X\rangle^n_d := \sum_{i=1}^n \langle Z_i, X_i\rangle_d. \tag{2.12}$$

We will use the notation $\langle Z, X\rangle = \langle Z, X\rangle^n_d$, when omitting the indices does not lead to confusion.

3. Optimal allocations of portfolios

To characterize Pareto-optimal allocations we describe, at first, allocations which minimize the total risk, i.e. solutions of the infimal convolution

$$\bigwedge \varrho_i(X) = \inf\left\{\sum_{i=1}^n \varrho_i(\xi_i) \mid (\xi_i) \in \mathcal{A}(X)\right\}. \tag{3.1}$$

The inf-convolution problem is a restricted optimization problem. It can be transformed into an unrestricted global minimization problem

$$\bigwedge \varrho_i(X) = \inf\left\{\overline{\varrho}(\xi) + 1_{\mathcal{A}(X)}(\xi) \mid \xi \in (L^p_d)^n\right\} \tag{3.2}$$

where $\overline{\varrho}(\xi) := \sum_{i=1}^n \varrho_i(\xi_i)$ and for a convex set A, $\mathbb{1}_A$ denotes the convex indicator

$$\mathbb{1}_A(x) = \begin{cases} 0, & x \in A, \\ \infty, & x \notin A. \end{cases} \tag{3.3}$$

We generally assume that there exists at least one n-allocation $\xi \in \mathcal{A}^n(X)$, where $\overline{\varrho}$ is continuous and finite. For $f : E \to \mathbb{R} \cup \{\infty\}$ the *domain of continuity* is denoted by

$$\operatorname{domc}(f) := \{x \in E \mid f \text{ is finite and continuous in } x\}.$$