Insurance: Mathematics and Economics 44 (2009) 409–414

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal allocation of policy limits and deductibles under distortion risk measures[☆]

Weiwei Zhuang, Zijin Chen, Taizhong Hu [*]

Department of Statistics and Finance, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

A R T I C L E I N F O

Article history:
Received August 2008
Received in revised form November 2008
Accepted 29 November 2008

MSC:
60E15

Subject Categories:
IM30
IE13

Keywords:
Comonotonicity
Distortion function
Loss frequency
Loss severity
Optimization criteria
Stochastic orders

A B S T R A C T

In the literature, orderings of optimal allocations of policy limits and deductibles were established by maximizing the expected utility of wealth of the policyholder. In this paper, by applying the bivariate characterizations of stochastic ordering relations, we reconsider the same model and derive some new refined results on orderings of optimal allocations of policy limits and deductibles with respect to the family of distortion risk measures from the viewpoint of the policyholder. Both loss severities and loss frequencies are considered. Special attention is given to the optimization criteria of the family of distortion risk measures with concave distortions and with only increasing distortions. Most of the results presented in this paper can be applied to some particular distortion risk measures. The results complement and extend the main results in Cheung [Cheung, K.C., 2007. Optimal allocation of policy limits and deductibles. Insurance: Mathematics and Economics 41, 291–382] and Hua and Cheung [Hua, L., Cheung, K.C., 2008a. Stochastic orders of scalar products with applications. Insurance: Mathematics and Economics 42, 865–872].

1. Introduction and motivation

Recently, Cheung (2007) and Hua and Cheung (2008a,b) considered the problem of optimal allocation of policy limits and deductibles from the viewpoint of a policyholder or an insurer. Their model is described as follows. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of n random losses (risks) faced by a policyholder, and let $\mathbf{S} = (S_1, \ldots, S_n)$ denote the vector of the occurrence times of the n insured risks, independent of $\mathbf{X}$. Then the total discounted loss of the policyholder is $\sum_{i=1}^{n} X_i e^{-\delta S_i}$, where δ is the discount rate. Through paying a premium P, the policyholder could obtain a total coverage (*policy limits* or *deductibles*) from an insurer with which she can allocate arbitrarily among the n risks. For a total of ℓ dollars as policy limit with premium $P(\ell)$, an allocation vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$ means that if the i risk occurs, then the policyholder will get $X_i \wedge \ell_i \equiv \min\{X_i, \ell_i\}$ right after the event of the loss from the insurer and the insurance coverage for this risk will terminate while the insurance coverage for other risks is still in effect. Here, of course, $\ell_i \geq 0$ for each i and $\sum_{i=1}^{n} \ell_i = \ell$. Let $\mathcal{A}(\ell)$ denote the class of all such allocation vectors $\boldsymbol{\ell}$. The total risk $T_{\mathbf{X},\mathbf{S}}(\boldsymbol{\ell})$ is captured by two components: the retained loss and the insurance premium; that is,

$$T_{\mathbf{X},\mathbf{S}}(\boldsymbol{\ell}) = \sum_{i=1}^{n} (X_i - X_i \wedge \ell_i) e^{-\delta S_i} + P(\ell)$$
$$= \sum_{i=1}^{n} (X_i - \ell_i)_+ e^{-\delta S_i} + P(\ell), \quad (1.1)$$

where $x_+ = \max\{x, 0\}$ for all x. Similarly, instead of policy limits, for a total of d dollars as policy deductible with premium $P(d)$, an allocation vector $\mathbf{d} = (d_1, \ldots, d_n)$ means that if the i risk occurs, then the policyholder will get $(X_i - d_i)_+$ right after the event of the loss from the insurer. Let $\mathcal{A}(d)$ denote the class of all such allocation vectors $\mathbf{d}$ with $d_i \geq 0$ for each i and $\sum_{i=1}^{n} d_i = d$. The total risk $R_{\mathbf{X},\mathbf{S}}(\mathbf{d})$ is now given by

$$R_{\mathbf{X},\mathbf{S}}(\mathbf{d}) = \sum_{i=1}^{n} [X_i - (X_i - d_i)_+]\, e^{-\delta S_i} + P(d)$$

☆ Supported by the National Natural Science Foundations of China (Nos 70821001 and 10771204), and the National Basic Research Program of China (973 Program, Grant No. 2007CB814901).

* Corresponding author.
E-mail address: thu@ustc.edu.cn (T. Hu).

0167-6687/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2008.11.012

$$= \sum_{i=1}^{n} (X_i \wedge d_i)\, \mathrm{e}^{-\delta S_i} + P(d). \tag{1.2}$$

From the viewpoint of a policyholder, a prudent risk management is to ensure that risk measures associated with $T_{\mathbf{X},\mathbf{S}}(\boldsymbol{\ell})$ and $R_{\mathbf{X},\mathbf{S}}(\mathbf{d})$ are as small as possible. This motivates us to consider the following two optimization criteria for seeking the optimal allocation vectors $\boldsymbol{\ell}^* = (\ell_1^*, \ldots, \ell_n^*)$ and $\mathbf{d}^* = (d_1^*, \ldots, d_n^*)$ such that

$$\varrho_h\text{-optimization:} \quad \varrho_h[T_{\mathbf{X},\mathbf{S}}(\boldsymbol{\ell}^*)] = \min_{\boldsymbol{\ell} \in \mathcal{A}(\ell)} \varrho_h[T_{\mathbf{X},\mathbf{S}}(\boldsymbol{\ell})] \tag{1.3}$$

and

$$\varrho_h\text{-optimization:} \quad \varrho_h[R_{\mathbf{X},\mathbf{S}}(\mathbf{d}^*)] = \min_{\mathbf{d} \in \mathcal{A}(d)} \varrho_h[R_{\mathbf{X},\mathbf{S}}(\mathbf{d})], \tag{1.4}$$

where $\varrho_h[\cdot]$ is the distortion risk measure with distortion function h. The formal definition of distortion risk measures is given in Section 2. Since the closed-form expressions of the optimal allocation vectors $\boldsymbol{\ell}^*$ and $\mathbf{d}^*$ are difficult to obtain, we turn to deriving useful qualitative results concerning the relative size of each allocation.

Cheung (2007) and Hua and Cheung (2008a) considered the following optimization problems for the above model:

$$\text{Problem L:} \quad \begin{cases} \max\limits_{\boldsymbol{\ell} \in \mathcal{A}(\ell)} \mathrm{E}\left[u\left(w - T_{\mathbf{X},\mathbf{S}}(\boldsymbol{\ell})\right)\right], \\ \text{where } u \text{ is increasing concave,} \\ \quad \text{and } w \text{ is a fixed constant;} \end{cases}$$

$$\text{Problem D:} \quad \begin{cases} \max\limits_{\mathbf{d} \in \mathcal{A}(d)} \mathrm{E}\left[u\left(w - R_{\mathbf{X},\mathbf{S}}(\mathbf{d})\right)\right], \\ \text{where } u \text{ is increasing concave,} \\ \quad \text{and } w \text{ is a fixed constant,} \end{cases}$$

where $u(\cdot)$ and w are the utility function and the wealth of the policyholder, respectively. In view of Proposition 2.4, Problems L and D are equivalent to ϱ_h-optimization problems (1.3) and (1.4) with concave distortion function h, respectively. Let $\boldsymbol{\ell}^*$ and $\mathbf{d}^*$ be the solutions to the ϱ_h-optimization problems (1.3) and (1.4) with concave distortion function h. Cheung (2007) proved that if $X_1, \ldots, X_n$ are independent and the discount rate $\delta = 0$ (without frequency impacts) then for each pair (i, j),

$$X_i \le_{\text{hr}} X_j \Longrightarrow \ell_i^* \le \ell_j^* \quad \text{and} \quad d_i^* \ge d_j^*, \tag{1.5}$$

and that if $X_1, \ldots, X_n$ are comonotonic (the *worst* dependence structure of severities of the risks) and $\delta = 0$ then

$$X_i \le_{\text{st}} X_j \Longrightarrow \ell_i^* \le \ell_j^* \quad \text{and} \quad d_i^* \ge d_j^*. \tag{1.6}$$

The formal definitions of the orders $\le_{\text{st}}, \le_{\text{hr}}$ and the orders $\le_{\text{lr}}, \le_{\text{rh}}$ that will appear below are given in Section 2. For details on the concept of comonotonicity, see Dhaene et al. (2002) and references therein. Hua and Cheung (2008a) proved that if $S_1, \ldots, S_n$ are independent, $X_1, \ldots, X_n$ are comonotonic and $\delta > 0$ then for each pair (i, j),

$$S_j \le_{\text{lr}} S_i,\ X_i \le_{\text{st}} X_j \Longrightarrow \ell_i^* \le \ell_j^* \quad \text{and} \quad d_i^* \ge d_j^*. \tag{1.7}$$

The purpose of this paper is to reconsider the above model and derive several refined results on this line, complementing and/or generalizing (1.5)–(1.7). More precisely, it is shown that:

- if $X_1, \ldots, X_n$ are independent and $\delta = 0$ then
  $$X_i \le_{\text{st}} X_j \Longrightarrow \ell_i^* \le \ell_j^*$$
  with respect to ϱ_h-optimization with concave distortion function h.
- if $X_1, \ldots, X_n$ are independent and $\delta = 0$ then
  $$X_i \le_{\text{st}} X_j \Longrightarrow d_i^* \ge d_j^*$$
  with respect to ϱ_h-optimization with convex distortion function h.
- if $X_1, \ldots, X_n$ are independent and $\delta = 0$ then
  $$X_i \le_{\text{rh}} X_j \Longrightarrow \ell_i^* \le \ell_j^*,$$
  $$X_i \le_{\text{hr}} X_j \Longrightarrow d_i^* \ge d_j^*,$$
  with respect to ϱ_h-optimization with any distortion function h.
- If $X_1, \ldots, X_n, S_1, \ldots, S_n$ are independent, then
  $$X_i \le_{\text{lr}} X_j,\ S_j \le_{\text{lr}} S_i \Longrightarrow \ell_i^* \le \ell_j^* \quad \text{and} \quad d_i^* \ge d_j^*$$
  with respect to ϱ_h-optimization with concave distortion function h.
- If $X_1, \ldots, X_n$ are comonotonic, and $S_1, \ldots, S_n$ are independent, then
  $$X_i \le_{\text{st}} X_j,\ S_j \le_{\text{rh}} S_i \Longrightarrow \ell_i^* \le \ell_j^* \quad \text{and} \quad d_i^* \ge d_j^*$$
  with respect to ϱ_h-optimization with concave distortion function h.

The rest of the paper is organized as follows. In Section 2, we recall the definitions of some stochastic orders and of Wang's distortion risk measure as well as some basic lemmas and facts that will be used in what follows. The main results on optimal allocation of policy limits and deductibles without frequency impacts and with frequency impacts are given in Sections 3 and 4, respectively. Section 5 concludes the paper, and Appendix presents the proofs of some lemmas appearing in Section 4.

Throughout, 'increasing' and 'decreasing' mean 'non-decreasing' and 'non-increasing', respectively. All expectations are implicitly assumed to exist when they are written.

2. Preliminaries

2.1. Stochastic orders

We recall from Shaked and Shanthikumar (2007) or Müller and Stoyan (2002) the definitions of some stochastic orders which will be used in this paper. Let X and Y be two random variables with distribution functions F and G, survival functions $\overline{F}$ and $\overline{G}$, respectively. We say that X is smaller than Y

- in the *usual stochastic order*, denoted by $X \le_{\text{st}} Y$, if $\overline{F}(t) \le \overline{G}(t)$ for all t or, equivalently, $\mathrm{E}[\phi(X)] \le \mathrm{E}[\phi(Y)]$ for all increasing functions ϕ;
- in the *hazard rate order*, denoted by $X \le_{\text{hr}} Y$, if $\overline{G}(t)/\overline{F}(t)$ is increasing in t;
- in the *reversed hazard rate order*, denoted by $X \le_{\text{rh}} Y$, if $G(t)/F(t)$ is increasing in t;
- in the *likelihood ratio order*, denoted by $X \le_{\text{lr}} Y$, if F and G have density or probability mass functions f and g, respectively, and if $g(t)/f(t)$ is increasing in t;
- in the *increasing convex [concave] order*, denoted by $X \le_{\text{icx}} [\le_{\text{icv}}]$ Y, if $\mathrm{E}[\phi(X)] \le \mathrm{E}[\phi(Y)]$ for all increasing convex [concave] functions ϕ.

In the literature of actuarial sciences, the order $\le_{\text{icx}}$ is also termed as the stop-loss order; see Denuit et al. (2005). The relationships among these orders are shown in the following diagram:

$$\begin{array}{ccccc} X \le_{\text{lr}} Y & \Longrightarrow & X \le_{\text{hr}} Y & & \\ \Downarrow & & \Downarrow & & \\ X \le_{\text{rh}} Y & \Longrightarrow & X \le_{\text{st}} Y & \Longrightarrow & X \le_{\text{icx}} Y \quad \text{and} \quad X \le_{\text{icv}} Y. \end{array}$$

The following two lemmas are useful in the proofs of the main results in Sections 3 and 4. For any bivariate function $g(x, y)$, define $\Delta g(x, y) = g(x, y) - g(y, x)$. Also define

$$\mathscr{G}_{\text{lr}} = \{g(x, y) :\ g(x, y) \ge g(y, x),\ \forall x \ge y\},$$
$$\mathscr{G}_{\text{hr}} = \{g(x, y) :\ \Delta g(x, y) \text{ is increasing in } x, \forall x \ge y\},$$
$$\mathscr{G}_{\text{rh}} = \{g(x, y) :\ \Delta g(x, y) \text{ is increasing in } x, \forall x \le y\},$$
$$\mathscr{G}_{\text{st}} = \{g(x, y) :\ \Delta g(x, y) \text{ is increasing in } x, \forall x\}.$$