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Risk factor contributions in portfolio credit risk models

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ABSTRACT

Determining contributions to overall portfolio risk is an important topic in risk management. For positions (instruments and sub-portfolios), this problem has been well studied, and a significant theory built, around the calculation of marginal contributions. We consider the problem of determining the contributions to portfolio risk of risk factors. This cannot be addressed through an immediate extension of techniques for position contributions, since the portfolio loss is a nonlinear function of the risk factors. We employ the Hoeffding decomposition of the portfolio loss into a sum of terms depending on the factors. This decomposition restores linearity, but includes terms arising from joint effects of groups of factors. These cross-factor terms provide information to risk managers, since they can be viewed as best hedges of the portfolio loss involving instruments of increasing complexity. We illustrate the technique on multi-factor portfolio credit risk models, where systematic factors represent industries, geographical sectors, etc.

1. Introduction

Decomposing portfolio risk into its different sources is a fundamental problem in financial risk management. Once a risk measure has been selected, and the risk of a portfolio has been calculated, a natural question to ask is: *where does this risk come from?* Specifically, the risk manager may be interested in understanding contributions to portfolio risk of two types:

- **Positions:** individual instruments, counterparties and sub-portfolios.
- **Risk factors:** various systematic or idiosyncratic factors affecting portfolio losses (e.g. market risk factors such as interest rates, exchange rates, equity volatilities etc., macro-economic, geographic, or industry factors affecting market or credit risk).

The development of methodologies for the first type of risk contribution is of great importance for hedging, capital allocation, performance measurement and portfolio optimization. In this case, the portfolio losses can be written as the sum of losses of individual positions (instruments, counterparties, sub-portfolios). For such sums, there is an established theory for *additive* risk contributions based on the concept of *marginal contributions*, sometimes referred to as *Euler allocation*. The latter name derives from the fact that the formula for contributions to risk measures which are homogeneous functions of degree one of the portfolio weights (standard deviation, VaR, CVaR, etc.) follows directly from Euler's theorem (see Tasche, 1999; Tasche, 2008). Position risk contributions in general, and the Euler allocation in particular, have received much attention in the recent literature. Koyluoglu and Stoker (2002) and Mausser and Rosen (2007) survey different contribution measures, with the latter reference focusing on applications to portfolio credit risk management. Kalkbrener (2005) considers a set of axioms for risk allocation methods, and demonstrates that they are only satisfied by the Euler allocation. Denault (2001) relates the Euler allocation to cooperative game theory and notions of fairness in allocation, while Tasche (1999) gives an interpretation in terms of optimization and performance measurement. Numerous authors have also considered computational issues arising in the calculation of position risk contributions. See, for example Emmer and Tasche (2005), Glasserman (2005), Glasserman and Li (2005), Huang et al. (2007), Kalkbrener et al. (2004), Merino and Nyfeler (2004), Tchistiakov et al. (2004) and Mausser and Rosen (2007) and the references therein.

Just as fundamental for risk management, the development of methodologies for contributions of *risk factors* to portfolio risk has received comparatively little attention. In this case, portfolio losses cannot generally be written as a linear function of the individual risk factors. When each position depends (perhaps in a nonlinear way) only on a single independent risk factor (or a small subset of risk factors), the problem can be addressed effectively

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by computing position contributions and transforming them to factor contributions. However, in many problems there are several factors which interact across large parts of the portfolio to drive potential losses, and the standard theory for deriving contributions cannot be directly applied. These factors might be systematic factors representing macro-economic variables[1], indices, or financial variables. Practical examples where such problems arise include:

- Portfolio credit risk, where multi-factor credit models are common.
- Portfolios with equity options (where equities are modelled using a multi-factor model), foreign exchange options or quanto options.
- Fixed income and interest rate derivatives portfolios.
- Collateralized Debt Obligations and Asset Backed Securities.

Contributions of risk factors are important because they facilitate an understanding of the sources of risk in a portfolio; this is especially important for complex portfolios with many instruments, where individual instrument risk contributions may not be too enlightening. It is also useful in understanding the sources of risk for complicated derivative securities (e.g. portfolio credit derivatives). Furthermore, the current credit crisis has highlighted the need for tools that help us understand better the role of systematic risk factors in credit risk.

Recent papers which consider the problem of factor contributions directly include the following. Tasche (2008) shows how the Euler allocation can be extended to calculate contributions of individual names to CDO tranche losses in a consistent manner and proposes measures for the impact of risk factors in the nonlinear case; Cherny and Madan (2007) study position contributions of conditional losses given the risk factors (see the discussion in Section 4 below); Rosen and Saunders (2009a) study the best hedge (in a quadratic sense) among linear combinations of the systematic factors in the context of the Vasicek model of portfolio credit risk; Cherny et al. (2008) consider the best approximation to portfolio losses by a sum of nonlinear functions of the individual factors, and present an explicit solution in the case of the Gaussian copula; Bonti et al. (2006) conceptualize the risk of credit concentrations as the impact of stress in one or more systematic risk factors on the loss distribution of a credit portfolio.

In this paper, we develop an extension of the Euler allocation that applies to nonlinear functions of a set of risk factors. The technique is based on the Hoeffding decomposition, originally developed for statistical applications (see, for example, van der Vaart, 1998; Sobol, 1993). The intuition behind the methodology is simple: while we cannot write the portfolio loss as a sum of functions of individual risk factors, the application of the Hoeffding decomposition allows us to express it as a sum of functions of all subsets of risk factors. The standard Euler allocation machinery can then be applied to the new loss decomposition. The price paid for this methodology is that we have to consider contributions not only from single risk factors, but also from the interaction of every possible collection of risk factors.

The terms in the Hoeffding decomposition also have an important risk management interpretation. They can be thought of as best-hedge portfolios of increasing complexity which depend on an increasing number of factors. Each term in the decomposition gives the best (quadratic) hedge of the residual risk in the portfolio that can be constructed with derivative securities depending on a given collection of factors, and which cannot be hedged by any subset of that collection (i.e. any previous term in the decomposition).

[1] See Bonfim (2009) for a recent study of the impact of macro-factors on credit risk.

For a portfolio with k risk factors, the Hoeffding decomposition contains 2^k terms. For a small number of factors, we can compute the contributions of all of the terms explicitly, and then perhaps aggregate them in financially meaningful ways. When the number of factors is large, it may be more practical to compute the contributions for a smaller set, plus a residual contribution. It is often the case that only a handful of risk factors are significant. Alternatively, one may aggregate the factors into a smaller number of subsets and assess their contributions.

We develop the methodology in detail in the context of multi-factor credit portfolio models and derive specific results for calculating systematic factor contributions. In this case, there are analytical expressions for conditional expectations of losses given subsets of systematic factors. Thus, the computational complexity of the methodology is equivalent to that of calculating standard position contributions. While there are analytical expressions for volatility contributions (although perhaps too complex for many practical applications), VaR/CVaR contributions present computational challenges. However, advanced methods developed for position contributions and cited above may be applicable to calculating factor contributions as well.

While the main focus of this paper is on credit risk, we stress that the use of the Hoeffding decomposition to compute contributions of risk factors, as developed in this paper, is general and can be applied to portfolio loss contributions in various financial problems, whether market, credit or operational risk. For example, alternative applications of the technique include the risk management of equity derivatives or interest rate derivatives portfolios. In the first case, the risk manager can compute the contributions to portfolio risk of all the major (macro-economic) industry sector and geographical factors driving equity returns as well as volatilities. Portfolios with complex basket options can easily be included. This leads to a better understanding of risk concentrations and potential macro-hedges of the portfolio. Similarly, the computation of factor contributions in a fixed-income derivatives portfolio (which may include multiple underlying risk-free spread curves) provides a clearer picture of systematic risk and possible ways of hedging or mitigating it.

Several papers have recently employed the Hoeffding decomposition in financial problems, although for different applications; Sobol (1993) has defined variance-based global sensitivity indices (see also Sobol and Kucherenko, 2005); Kucherenko and Shah (2007) consider applications to Monte-Carlo methods for options pricing; Lemieux and L'Ecuyer (2001) show how to use the decomposition to develop selection criteria for Quasi-Monte Carlo point sets, and present an application to derivatives pricing; Baur et al. (2004) use the decomposition to consider the sensitivity of portfolio credit risk models to model risk (mis-specification of model parameters and dependence structure) in a variance-based context.

The remainder of the paper is structured as follows: Section 2 discusses factor models for credit risk, setting notation and reviewing results that will be used in later sections. Section 3 briefly surveys the theory of risk contributions for positions, emphasizing marginal risk contributions and the Euler allocation principle. Section 4 presents Hoeffding decompositions and discusses their use to calculate factor contributions to portfolio credit risk, as well as various extensions and practical issues that arise when using the technique. Section 5 presents numerical examples applying the theory of factor contributions from Section 4 to sample portfolios. Section 6 concludes.

2. Factor models for credit risk

We denote the portfolio loss function by L, and assume it can be written as the sum of losses due to individual positions,