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Optimal allocation of change points in simple step-stress experiments under Type-II censoring

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ABSTRACT

In simple step-stress experiments under Type-II censoring with the cumulative exposure model and exponentially distributed lifetimes, maximum likelihood estimates (MLE) of the expected lifetimes may not exist due to the absence of failure times either before or after the stress change point. For this reason, when planning a step-stress experiment, the change point could be chosen so as to minimize the probability of non-existence of the MLE. These non-existence probabilities are examined and compared in the one- as well as the two-sample situations. Moreover, the optimal allocations of the change points are discussed and the effects of the use of non-optimal choices for the change points are assessed.

1. Introduction

In accelerated life testing (ALT) applications, in order to speed up the testing procedure and to reduce the time to failure, the experimental units are exposed to higher stress levels than normal. The data from such an ALT are then transformed to estimate the distribution of failures under normal operating condition. Primary references on ALT are (among others) Nelson (1990), Meeker and Escobar (1998) and Bagdonavicius and Nikulin (2002).

One popular ALT model is the step-stress model, where the stress put on the testing units is not constant and is varied at two or more ordered levels. The simple step-stress model, which plays a notable role in accelerated life testing, consists of two stages and allows for testing under two different conditions at these stages of the experiment (usually referred to as stress levels). Interest lies on inference for the expected mean lifetimes θ_1 and θ_2 at these stress levels, when the underlying distributions are exponential. For details pertinent to inferential and optimization aspects in the simple step-stress model under Type-II censoring for exponentially distributed lifetimes, we refer to Balakrishnan et al. (2007), Bai et al. (1989), and Miller and Nelson (1983). Wang (2009) worked under the assumptions of the exponential step-stress model, Type-I or Type-II censored, while Abdel-Hamid and Al-Hussaini (2009) considered a more general life distribution under Type-I censoring. A detailed survey of all developments on step-stress models has been provided by Balakrishnan (2009). Recently, Kateri et al. (2009) introduced a meta-analysis approach for multiple step-stress experiments. Under this more general model, s ($s \geq 2$) independent step-stress experiments are carried out for the same population and under the same stress levels. What can vary among the s samples, thus giving a flexibility from a design point of view, are their sizes, the proportion of censoring in each of these samples, and the time points at which the stress levels are changed.

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The multi-sample step-stress set-up may, therefore, be viewed as a framework to analyze different experiments in the sense of a meta-analysis as well as the basis to plan experiments in a reasonable or, even better, in an optimal way. Here, we tackle this specific problem and discuss the optimal allocation of change points for the characteristic case of simple step-stress experiments.

In this paper, we focus on the special case of two independent samples and study its features, along with those of the one-sample simple step-stress model in the context of designing an ALT experiment. In terms of experimental design, it makes sense to consider a one-sample experiment with a sample of n units and a total of r observed failures as opposed to a two-sample experiment with sample sizes n_1, n_2 and corresponding number of failures being r_1, r_2 such that $n_1 + n_2 = n$ and $r_1 + r_2 = r$. Given the total sample size n, which is often fairly small, a choice of $s > 2$ sub-samples may be unrealistic. However, the effects and issues pointed out here in the two-sample case remain valid and the findings may be extended to this general case as well.

One problem with step-stress models is that the MLE may not always exist. This happens when there is no observation under some particular stress level. In the one- and two-sample cases, we derive and analyze the probabilities of non-existence of the MLE and then discuss optimal allocations of the change points by minimizing the non-existence probabilities of the MLE of the parameters θ_1 and θ_2.

Optimal design of step-stress tests could also be based on other criteria chosen by the experimenter. Standard criteria are Var-optimality and D-optimality, that minimize the asymptotic variance of the MLE of the mean life at a specified design stress or the determinant of the information matrix, respectively. However, since existence of the MLEs is of particular interest, we focus here on minimizing the non-existence probabilities of the MLEs. In a general step-stress set-up, due to the large number of parameters, it will not be possible to just provide tables of respective optimal plans even in one-criteria situations. A careful consideration and study of several plots as those presented in the subsequent sections will be necessary to make concrete decisions.

The structure of the paper is as follows. In Section 2, the two-sample experiment is introduced along with the MLE in the one- and the two-sample set-ups. In Section 3, the probabilities of non-existence of the MLE of θ_1 and θ_2 are derived for the one- and the two-sample experiments, and these probabilities are then examined, compared and interpreted. Finally, in Section 4, we address the optimal allocation of the change points and point out the importance of a good choice of the change points in the design of such a step-stress experiment.

2. Preliminaries

We restrict ourselves here to introducing the notation in the two-sample case, since the corresponding one for the multi-sample case becomes apparent. Let two independent samples of sizes n_1 and n_2 be placed on a life test, subjected to an initial stress level of x_1. During the experimental period, the stress level is increased to x_2 at pre-fixed change point τ_k for the kth sample, for $k = 1, 2$. Moreover, under Type-II censoring, the experiment is terminated when a pre-fixed number r_k ($r_k \le n_k$) of failures is observed in the kth sample ($k = 1, 2$). Thus, the total number of failures in the whole experiment is $r = r_1 + r_2$. The order statistics which describe the failures in the kth sample are denoted by $T^{(k)}_{1:n_k}, \ldots, T^{(k)}_{r_k:n_k}$, with given sample size n_k, and the corresponding observations by $\left(t^{(k)}_{i:n_k}\right)_{1\le i\le r_k}$ for $k = 1, 2$.

The numbers of failures at the first stress level are given by

$$R_{1k} = \#\{i : T^{(k)}_{i:n_k} \le \tau_k\},$$

with $0 \le R_{1k} \le r_k$, $k = 1, 2$.

We assume that the observed ordered failure times $t^{(k)}_{i:n_k}$, $1 \le i \le r_k$, come from a cumulative exposure model (cf. Nelson, 1990). Under this model, it is assumed that the residual lifetimes of the experimental units depend only on the cumulative exposure of the units without any memory of how the accumulation of the exposure occurred. In this case, the CDF of the lifetime of a test unit under the simple step-stress model is given by

$$F_{k,\theta_1,\theta_2}(t) = \begin{cases} F_{1k}(t), & 0 < t < \tau_k \\ F_{2k}(c_k + t - \tau_k), & \tau_k \le t < \infty, \end{cases} \tag{2.1}$$

for $k = 1, 2$, where $F_{jk}(x)$ is the CDF of the life distribution of the kth sample under the jth ($j = 1, 2$) stress level. The translation factor c_k is the solution of the equation

$$F_{2k}(c_k) = F_{1k}(\tau_k), \quad k = 1, 2. \tag{2.2}$$

If we additionally assume exponential lifetimes, then $F_{jk}(t)$ in (2.1) is the CDF of the exponential distribution $\mathrm{Exp}(\theta_j)$, $j = 1, 2$, i.e., $F_{jk}(t) = 1 - \mathrm{e}^{-\frac{t}{\theta_j}}$, and the solutions of (2.2) turn out to be

$$c_k = \tau_k \frac{\theta_2}{\theta_1}, \quad k = 1, 2. \tag{2.3}$$