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# Excess based allocation of risk capital☆

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ABSTRACT

In this paper we propose a new rule to allocate risk capital to portfolios or divisions within a firm. Specifically, we determine the capital allocation that minimizes the excesses of sets of portfolios in a lexicographical sense. The excess of a set of portfolios is defined as the expected loss of that set of portfolios in excess of the amount of risk capital allocated to them. The underlying idea is that large excesses are undesirable, and therefore the goal is to determine the allocation for which the largest excess is as small as possible. We show that this allocation rule yields a unique allocation, and that it satisfies some desirable properties. We also show that the allocation can be determined by solving a series of linear programming problems.



## 1. Introduction

Regulators require that in order to be able to hold a risky position, financial institutions withhold a level of capital, referred to as *risk capital*. The risk capital needs to be added to the risky position and invested safely in order to act as a buffer and reduce the adverse effects of unfavorable events on the solvency of the firm. The focus in this paper is on the allocation of the total amount of risk capital to different subportfolios, divisions, or lines of business. As argued by, e.g., Tasche (1999), withholding risk capital is costly, and therefore allocating risk capital to individual investments is important for performance evaluation as well as for pricing decisions. The allocation problem is nontrivial because the amount of risk capital allocated to a portfolio consisting of multiple subportfolios is typically less than the sum of the amounts of risk capital that would need to be withheld for each subportfolio separately. The underlying intuition is that because the risks of different subportfolios are typically not perfectly correlated, some hedge potential may arise from combining different subportfolios. The issue then is what is a "fair" division of these diversification gains over the subportfolios.

The allocation problem described above has received considerable attention in the literature. Tasche (1999) considers allocation of risk capital to financial instruments in a portfolio, and argues that the only "appropriate" way to allocate risk capital for performance measurement purposes is to determine the *marginal risk contribution* of each investment. The marginal risk contribution is defined as the derivative of the aggregate risk capital with respect to the weight of the financial instrument in the portfolio. Denault (2001) instead proposes a game-theoretic approach to determine risk capital allocations for companies with multiple business divisions. He focuses on risk capital allocations that are "fair" in the sense that no set of divisions is allocated more risk capital than the amount of risk capital that they would need to withhold if they were on their own. He shows that when business divisions are infinitely divisible, the only allocation that satisfies this fairness condition is the marginal risk contribution defined above. In game-theoretic terms, this allocation is referred to as the Aumann–Shapley value. For the special case where the risk measure is Expected Shortfall, the corresponding allocation rule is also referred to as the Conditional Tail Expectation (CTE) rule (see, e.g., Overbeck, 2000; Panjer, 2002, and Dhaene et al., 2008, 2009).

The literature on risk capital allocation has then evolved in several directions. First, there is some literature that considers the risk capital allocation that results from using the Aumann–Shapley

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value when a specific risk measure is used (Tsanakas and Barnett, 2003; Tsanakas, 2009), or when the portfolio losses have a specific probability distribution (Panjer, 2002, and Landsman and Valdez, 2003). Second, there is some literature that focuses on generalizations or extensions of the Aumann–Shapley value (e.g. Fischer, 2003; Tsanakas, 2004; Powers, 2007; Furman and Zitikis, 2008), or on capital allocations that result when alternative game-theoretic concepts are used (Csòka, 2008). Third, Myers and Read (2001), Sherris (2006), and Kim and Hardy (2009) consider alternative capital allocation rules based on solvency ratios or expected return. Finally, there is a stream of literature in which the capital allocation is determined as the solution of an optimization problem (Dhaene et al., 2003; Laeven and Goovaerts, 2004; Goovaerts et al., 2005, and Dhaene et al., 2009). Specifically, they consider capital allocations such that the (weighted) sum of a measure for the deviation of the business unit's losses from its allocated risk capital is minimized. Dhaene et al. (2009) show that by choosing specific deviation measures and/or specific weights, one can reproduce several of the allocation techniques proposed in the literature, including, e.g., the CTE rule.

In this paper we propose an alternative approach to allocate risk capital that falls into the latter stream of literature. Our approach is inspired by the fact that allocating risk capital on the basis of the Aumann–Shapley value can lead to undesirable allocations in the sense that the expected *excess loss*, i.e., the loss of the subportfolio in excess of the amount of risk capital allocated to it, can differ substantially across subportfolios. Large differences in expected excess losses could be perceived as unfair by managers who are evaluated based on the risk capital allocated to their portfolios. Therefore, we propose an alternative allocation rule in which the goal is to determine the capital allocation that minimizes the excesses of all subportfolios in a lexicographical sense. This implies that, from a set of feasible allocations, one first selects those allocations for which the highest excess is minimized. Within the set of allocations for which the highest excess is minimized, one then determines those allocations for which the second highest excess is minimized, and so on. This approach differs from the existing optimization approaches in two ways. First, whereas the existing literature focuses on minimizing the aggregate (weighted) excess over all portfolios, we consider each excess separately. Second, we do not only take into account the excesses of the individual portfolios, but also of all possible subsets of portfolios.

The paper is organized as follows. In Section 2 we formally define risk capital allocation problems, and show that the Aumann–Shapley value can lead to allocations in which the excess losses are significantly different across portfolios. We then define the alternative risk capital allocation rule that we propose, which we will refer to as the Excess Based Allocation (EBA). In Section 3 we define desirable properties of a risk capital allocation rule, and show that EBA satisfies these properties. In Section 4 we show how EBA can be determined by solving a sequence of linear programming problems. In Section 5 we apply EBA to the capital allocation problem of a life insurance company, and compare the result to capital allocations resulting from other capital allocation rules described in the literature. We conclude in Section 6.

## 2. Model

In this section, we first define risk capital allocation problems and risk capital allocation rules. Next, we discuss an allocation rule that has received considerable attention in the literature. We then propose an alternative risk capital allocation rule.

### 2.1. Risk capital allocation problems

Our focus in this paper is on the allocation of risk capital to subportfolios. We consider a portfolio consisting of $n$ subportfolios, indexed by $i = 1, \dots, n$. Subportfolio $i$ generates a random loss $X_i$ at a given future date, and so the aggregate loss is given by $\sum_{i=1}^{n} X_i$. Regulators require that, in order to be able to hold this risky position, an amount of risk capital should be withheld and invested safely. The total required amount of risk capital is determined by means of a risk measure, and the issue is how to allocate this total amount to the $n$ subportfolios.

Throughout this paper we will use this following notation:

- $N = \{1, \dots, n\}$ denotes the set of subportfolios;
- $\Omega = \{\omega_1, \dots, \omega_m\}$ denotes the finite set of states of the world;
- $\pi(\omega) > 0$ denotes the probability that state $\omega \in \Omega$ occurs[1];
- $X_i(\omega)$ denotes the loss from subportfolio $i$ in state $\omega \in \Omega$;
- $X = \{X_1, \dots, X_n\}$ denotes the losses of the $n$ subportfolios;
- $\mathcal{V}$ denotes the set of random variables on $\Omega$;
- $\mathbf{1} \in \mathcal{V}$ denotes the random variable given by: $\mathbf{1}(\omega) = 1$ for all $\omega \in \Omega$.

The amount of risk capital that needs to be withheld in order to be able to hold a risky position $Y \in \mathcal{V}$ is determined by means of a risk measure $\rho : \mathcal{V} \to \mathbb{R}$. Following Artzner et al. (1999), we consider the case where risk capital is determined by means of a coherent risk measure. A risk measure $\rho : \mathcal{V} \to \mathbb{R}$ is coherent if and only if it satisfies the following four properties:[2]

(i) *Subadditivity*: for all $Y_1, Y_2 \in \mathcal{V}$

$$\rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2).$$

(ii) *Monotonicity*: for all $Y_1, Y_2 \in \mathcal{V}$ such that $Y_1(\omega) \geq Y_2(\omega)$ for all $\omega \in \Omega$

$$\rho(Y_1) \geq \rho(Y_2).$$

(iii) *Positive Homogeneity*: for all $Y \in \mathcal{V}$ and $c \geq 0$

$$\rho(c \cdot Y) = c \cdot \rho(Y).$$

(iv) *Translation Invariance*: for all $Y \in \mathcal{V}$ and $c \in \mathbb{R}$

$$\rho(Y + c \cdot \mathbf{1}) = \rho(Y) + c.$$

While all the results presented in this paper hold true for any coherent risk measure, in our numerical examples we will use Expected Shortfall as risk measure. In the literature, Expected Shortfall has been defined in different ways and not all definitions result in a coherent risk measure. This is the case in particular if the distribution of the loss has point masses. To ensure coherence, we use the definition which is due to Acerbi and Tasche (2002). First, let $q_{1-\alpha}(Y)$ denote the upper $100 \cdot (1-\alpha)$%-quantile of $Y \in \mathcal{V}$, i.e.,

$$q_{1-\alpha}(Y) = \inf\{y \in \mathbb{R} | P(Y \leq y) > 1 - \alpha\}. \tag{1}$$

Then, Expected Shortfall is defined as[3]

$$\rho_\alpha^{ES}(Y) = \frac{E(Y \cdot 1_{Y \geq q_{1-\alpha}(Y)}) - q_{1-\alpha}(Y) \cdot [P(Y \geq q_{1-\alpha}(Y)) - \alpha]}{\alpha}. \tag{2}$$

1 We assume that for each state $\omega \in \Omega$, it holds that $\pi(\omega) > 0$, i.e., the probability that the state occurs is strictly positive. Because we consider a finite state space, this is without loss of generality; states that occur with zero probability can be omitted from the state space.

2 Note that whereas Artzner et al. (1999) considers random gains, $X \in \mathcal{V}$ in our case is defined as a loss. This affects the *Monotonicity* and the *Translation Invariance* property.

3 If the losses have a continuous distribution, (2) simplifies to $\rho_\alpha^{ES}(Y) = E[Y | Y \geq q_{1-\alpha}(Y)]$. However, if $Y$ is not continuous, this does not yield a coherent risk measure; more specifically, *Subadditivity* is violated.