Insurance: Mathematics and Economics 49 (2011) 310–324

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Asymptotics for risk capital allocations based on Conditional Tail Expectation

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ARTICLE INFO

Article history:
Received February 2011
Received in revised form
May 2011
Accepted 10 May 2011

MSC:
primary 91B30
secondary 91G10
62H20
60E05

Keywords:
Asymptotic dependence and independence
Capital allocation
Conditional Tail Expectation
Extreme Value Theory
Heavy-tailed distributions
Value-at-Risk

ABSTRACT

An investigation of the limiting behavior of a risk capital allocation rule based on the Conditional Tail Expectation (CTE) risk measure is carried out. More specifically, with the help of general notions of Extreme Value Theory (EVT), the aforementioned risk capital allocation is shown to be asymptotically proportional to the corresponding Value-at-Risk (VaR) risk measure. The existing methodology acquired for VaR can therefore be applied to a somewhat less well-studied CTE. In the context of interest, the EVT approach is seemingly well-motivated by modern regulations, which openly strive for the excessive prudence in determining risk capitals.

1. Introduction

Let X denote an insurance risk. Speaking more formally, X is a non-negative random variable defined on the probability space $(\Omega, \mathcal{F}, \Pr)$ and possessing a distribution function $F(x) := \Pr(X \leq x)$ and a tail function $\overline{F}(x) := 1 - F(x)$, $x \in \mathbf{R}$. A risk measure is generally formulated as a functional, Q, from the space of distribution functions to $[0, \infty]$. Similarly (see, e.g. Bühlmann, 1980), we can consider the functional Q as from $\mathcal{X}$, the space of insurance risks, to $[0, \infty]$, which we indeed often do in the sequel.

Certainly, $\overline{F}$ establishes a meaningful ordering of $\mathcal{X}$ and, hence, it can be interpreted as a risk measure. However, for the sake of risk capital determination, it is desirable for the risk measure $Q[F]$ to take on monetary units. Thus, $Q[F] = \overline{F}(x)$ is naturally replaced with, e.g., its inverse, bringing us to the notion of Value-at-Risk (VaR). Namely, let $q \in (0, 1)$ denote the confidence level required by regulations. Then

$$\mathrm{VaR}_q[X] := \inf\{x : F(x) \geq q\}$$

establishes arguably the most popular risk measure, which has been a cornerstone of the financial risk measurement of the last century. We note in passing that the Solvency II Accord designed by the EU Commission sets $q = 0.995$ over a one-year time horizon.

Noticeably, the recent financial instability and, as a result, regulators' inclination to excessive prudence in determining risk capital requirements have to a certain extent enfeebled VaR's status. In this respect, the so-called tail-based risk measurement has emerged as a natural tool for quantifying insurance risks while emphasizing the adverse effect of low probability but high severity tail events. Thereby, a more pessimistic Conditional Tail Expectation (CTE) risk measure is defined, for $q \in (0, 1)$, as

$$\mathrm{CTE}_q[X] := \mathbf{E}[X | X > \mathrm{VaR}_q[X]].$$

CTE is known as a coherent risk measure over the space of continuous random variables. It belongs to both the distorted and weighted risk measures (see Wang, 1996; Dhaene et al., 2006; Furman and Zitikis, 2008a). Practically, CTE has already replaced VaR in regulatory requirements of, e.g., Canada, Israel

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doi:10.1016/j.insmatheco.2011.05.002

and Switzerland. We note in passing that the current practice in the aforementioned countries is $q = 0.99$ over a one-year time horizon.

Let $X_i \in \mathcal{X}$, $i = 1, \ldots, d$, denote $d \in \mathbf{N}$ insurance risks and let $S_d := X_1 + \cdots + X_d$ denote the aggregate risk. Then evaluating $\mathrm{VaR}_q[S_d]$ and $\mathrm{CTE}_q[S_d]$ is a somewhat basic phase of the modern risk capital framework. Indeed, while it is of pivotal importance to determine the overall risk capital requirement for an insurance company, it is of consequent interest to decompose the aforementioned capital into the associated risk sources. To this end, the functional Q is naturally generalized beyond the conditional state independence, to a risk capital allocation functional, A, from the space of the Cartesian product of $\mathcal{X}$ with itself to $[0, \infty]$, and such that $A[X_i, X_i] = Q[X_i]$, $i = 1, \ldots, d$; see, e.g., Furman and Zitikis (2008b). It should be noted that apart from purely regulatory interest, the functional A is often employed for, e.g., profitability analysis, pricing and quality control.

Various functional forms of A have been proposed in the literature, with the allocation based on the CTE risk measure, formulated as

$$\mathbf{E}[X_i|S_d > \mathrm{VaR}_q[S_d]], \quad i = 1, \ldots, d, \tag{1.1}$$

being arguably the most popular. See Section 6.3 of McNeil et al. (2005) for related discussions on this allocation as a consequence of the Euler principle, as well as Dhaene et al. (2011) for optimality studies of interest. Although (1.1) is quite elegant and satisfies many desirable properties, its analytic tractability for generally distributed and possibly dependent $X_1, \ldots, X_d$ remains seldom feasible. To emphasize the point, we refer the reader to Panjer and Jia (2001), Landsman and Valdez (2003), Valdez and Chernih (2003), Cai and Li (2005), Furman and Landsman (2005, 2006, 2008), Chiragiev and Landsman (2007), Vernic (2006, 2011) and Dhaene et al. (2008) for analytic expressions for (1.1) under specific multivariate distributions of a multi-line business and/or a portfolio of risks.

In this paper, we follow a different route. Namely, as the excessive prudence of the current regulatory framework requires a confidence level close to 1, the notion of Extreme Value Theory (EVT) becomes appropriate. We therefore study the asymptotic behavior of capital allocations defined in (1.1) as $q \uparrow 1$, when $X_1, \ldots, X_d$ are asymptotically dependent or asymptotically independent. Following Section 5.2 of McNeil et al. (2005), the asymptotic independence between two random variables X_i and X_j with distribution functions F_i and F_j is defined as

$$\lim_{q\uparrow 1} \Pr\left(F_j(X_j) > q|F_i(X_i) > q\right) = 0,$$

while the asymptotic dependence is defined via this relation with a positive limit. However, in this paper we slightly relax the notion of asymptotic dependence and define it as

$$\liminf_{q\uparrow 1} \Pr\left(F_j(X_j) > q|F_i(X_i) > q\right) > 0. \tag{1.2}$$

The notion of asymptotic dependence in higher dimensions is an obvious generalization of the two-dimensional definition above. It is known that, for both Fréchet and Gumbel cases, the CTE and VaR of a single risk are proportional for a high confidence level (see Asimit and Badescu, 2010). Therefore, not surprisingly our main results show that capital allocations, as described by (1.1), are asymptotically proportional to $\mathrm{VaR}_q[S_d]$ with a readily calculable coefficient of proportionality. This allows for the utilization of the VaR-related machinery when dealing with the risk capital allocation based on CTE.

The remainder of the paper is organized as follows. The main results under asymptotic dependence and asymptotic independence are formulated and proved in Sections 2 and 3, respectively. Relevant examples are discussed in Section 4, while certain simulation studies verifying the accuracy of the main results are carried out in Section 5. Finally, Section 6 concludes the paper.

2. Main results under asymptotic dependence

From now on, we consider a multi-line insurance business consisting of d non-negative risk variables $X_1, \ldots, X_d$. Denote $\underline{X} = (X_1, \ldots, X_d)$ and $S_d = \sum_{i=1}^d X_i$. Unless otherwise stated, all limit relationships hold as $t \to \infty$ or $q \uparrow 1$, letting the relations speak for themselves. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(\cdot) \sim cb(\cdot)$ to mean strong equivalence, i.e., $\lim a(\cdot)/b(\cdot) = c$ for some positive constant c, and we write $a(\cdot) \asymp b(\cdot)$ to mean weak equivalence, i.e., $0 < \liminf a(\cdot)/b(\cdot) \le \limsup a(\cdot)/b(\cdot) < \infty$. We also denote $\liminf a(\cdot)/b(\cdot) \ge 1$ and $\limsup a(\cdot)/b(\cdot) \le 1$ by $a(\cdot) \gtrsim b(\cdot)$ and $a(\cdot) \lesssim b(\cdot)$, respectively.

To develop the main results of this paper we extensively employ the EVT techniques. A distribution function F is said to belong to the *Maximum Domain of Attraction* (MDA) of a non-degenerate distribution function G, written as $F \in \mathrm{MDA}(G)$, if there are some $a_n > 0$ and $b_n \in \mathbf{R}$ for $n \in \mathbf{N}$ such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x).$$

Due to the Fisher–Tippett theorem (see Fisher and Tippett, 1928; Gnedenko, 1943), G is of one of the following three types:

$$\text{Fréchet type: } \Phi_\alpha(x) = \exp\left\{-x^{-\alpha}\right\}, \quad x > 0,\ \alpha > 0,$$

$$\text{Gumbel type: } \Lambda(x) = 1 - \exp\left\{-\mathrm{e}^{-x}\right\}, \quad -\infty < x < \infty,$$

$$\text{Weibull type: } \Psi_\alpha(x) = \exp\left\{-(-x)^\alpha\right\}, \quad x \le 0,\ \alpha > 0.$$

Since distributions from $\mathrm{MDA}(\Psi_\alpha)$ have finite upper endpoints while we are interested in risk variables with unbounded supports, in this paper we shall consider the Fréchet and Gumbel cases only.

Another important notion that is crucial for establishing our main results is *vague convergence*. Let $\{\mu_n, n \ge 1\}$ be a sequence of measures on a locally compact Hausdorff space $\mathbb{B}$ with countable base. Then μ_n converges vaguely to some measure μ, written as $\mu_n \xrightarrow{v} \mu$, if for all continuous functions f with compact support we have

$$\lim_{n\to\infty} \int_{\mathbb{B}} f \,\mathrm{d}\mu_n = \int_{\mathbb{B}} f \,\mathrm{d}\mu.$$

A thorough background on vague convergence is given by Kallenberg (1983) and Resnick (1987).

2.1. Fréchet case

The next assumption is sufficient for our first main result.

Assumption 2.1. Let $\underline{X}$ be a non-negative random vector with marginal distributions $F_1, \ldots, F_d$ such that

$$\lim_{t\to\infty} \frac{\Pr(X_1 > tx_1, \ldots, X_d > tx_d)}{\bar{F}_1(t)} := H_F(\underline{x})$$

exists for all $\underline{x} = (x_1, \ldots, x_d) \in [0, \infty]^d \setminus \{\underline{0}\}$, where $H_F(\cdot)$ is assumed to be a non-degenerate function and $\underline{0}$ is the vector of zeroes.

This assumption implies that the marginal distribution functions are tail equivalent. That is,

$$0 < \lim_{t\to\infty} \frac{\bar{F}_j(t)}{\bar{F}_i(t)} < \infty, \tag{2.1}$$

for all $1 \le i, j \le d$. In addition, there exists some $\alpha > 0$ such that, for each $1 \le i \le d$, the distribution function of X_i is regularly varying with index α, written as $X_i \in \mathcal{R}_{-\alpha}$,

$$\lim_{t\to\infty} \frac{\bar{F}_i(tx)}{\bar{F}_i(t)} = x^{-\alpha}, \quad x > 0. \tag{2.2}$$