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Risk capital allocation and cooperative pricing of insurance liabilities

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Abstract

The Aumann–Shapley [Values of Non-atomic Games, Princeton University Press, Princeton] value, originating in cooperative game theory, is used for the allocation of risk capital to portfolios of pooled liabilities, as proposed by Denault [Coherent allocation of risk capital, J. Risk 4 (1) (2001) 1]. We obtain an explicit formula for the Aumann–Shapley value, when the risk measure is given by a distortion premium principle [Axiomatic characterisation of insurance prices, Insur. Math. Econ. 21 (2) (1997) 173]. The capital allocated to each instrument or (sub)portfolio is given as its expected value under a change of probability measure. Motivated by Mirman and Tauman [Demand compatible equitable cost sharing prices, Math. Oper. Res. 7 (1) (1982) 40], we discuss the role of Aumann–Shapley prices in an equilibrium context and present a simple numerical example.

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1. Introduction

We discuss the problem of allocating aggregate capital requirements for a portfolio of pooled liabilities to the instruments that the portfolio consists of. Cooperative game theory (Shapley, 1953; Aumann and Shapley, 1974; Aubin, 1981) provides a suitable framework for cost allocation problems (e.g. Lemaire, 1984). Typically, provisions are made so that the allocation of total costs does not produce disincentives for cooperation to any player (instrument) in the game or coalition of players (sub-portfolio of instruments). A closely related approach has been to determine an allocation scheme by imposing economically motivated axioms on the system of prices that it would produce (Billera and Heath, 1982; Mirman and Tauman, 1982). The solution concept that emerges from the latter approaches is the celebrated Aumann and Shapley (1974) value. In the context of capital allocation, cooperation can be understood as the pooling of risky portfolios, and the cost as the 'risk capital' that a regulator decides that the holder of the

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portfolios should carry. This case was discussed in depth by Denault (2001), who used a cost functional based on a coherent risk measure (Artzner et al., 1999), and the Aumann–Shapley value emerged again as an appropriate solution concept. Explicit calculations of the Aumann–Shapley value were provided when the risk measure used is Expected Shortfall (this problem was solved by Tasche (2000a) in the context of performance measurement) and for the case of a risk measure used by the Securities and Exchange Commission.

In this paper we calculate an analytic formula for the Aumann–Shapley value using quantile derivatives (Tasche, 2000b), for the case that the risk measure belongs to the class of distortion principles (Denneberg, 1990; Wang et al., 1997). We obtain a representation of the Aumann–Shapley value, i.e. the capital allocated to each portfolio, as the expected value of the portfolio under a change of probability measure. This representation creates a formal link between problems of allocating capital and pricing risks. We discuss this relationship through the example of a pool, which covers specific liabilities carried by a number of insurers, who in turn make cash contributions to the pool, that can be interpreted as risk premia.

It was shown by Mirman and Tauman (1982) that the Aumann–Shapley value yields equilibrium prices in a monopolistic production economy. Motivated by this work, we generalise the example to a case where the different insurers choose the extent of coverage received from the pool, by expected utility maximisation. This set-up is quite different from the equilibrium models usually found in the literature on risk sharing, for example, Borch (1962), Bühlmann (1980), Taylor (1995), Aase (2002). In these papers, market prices are obtained via a clearing condition, which is not applicable to the problem that we discuss. Finally, we provide a simple numerical example, where the pool offers stop-loss protection to the participating insurers.

2. Coherent risk measures and distortion principles

A coherent risk measure is defined by Artzner et al. (1999) as a functional $\rho(X)$ on a collection of random cashflows (in our case X will be a non-negative random variable representing liabilities) that satisfies the following properties:

- Monotonicity: If $X \leq Y$ a.s. then $\rho(X) \leq \rho(Y)$,
- Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$,
- Positive homogeneity: If $a \in \mathbb{R}_+$ then $\rho(aX) = a\rho(X)$,
- Translation invariance: If $a \in \mathbb{R}$ then $\rho(X+a) = \rho(X) + a$.

$\rho(X)$ is interpreted as “the minimum extra cash that the agent has to add to the risky position X, and to invest ‘prudently’, to be allowed to proceed with his plans” (Artzner et al., 1999). ‘Invest prudently’, in this paper, means with zero interest.

All functionals satisfying the above properties allow a representation:

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}[X], \tag{1}$$

where $\mathcal{P}$ is a collection of probability measures.

Two random variables X, Y are called comonotonic if there is a random variable U and non-decreasing functions g, h such that $X = g(U)$, $Y = h(U)$ (Denneberg, 1994a). Comonotonicity corresponds to the strongest form of positive dependence between random variables. An additional desirable property of risk measures is additivity for comonotonic risks:

$$\text{Comonotonic additivity:} \quad \text{If } X, Y \text{ comonotonic then } \rho(X+Y) = \rho(X) + \rho(Y).$$

It can be shown that, if and only if $\rho(X)$ is a coherent risk measure satisfying comonotonic additivity, it has a representation as a Choquet integral with respect to a submodular set function or capacity, v (Choquet, 1953;