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Risk capital decomposition for a multivariate dependent gamma portfolio

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Abstract

This paper examines the tail conditional expectation risk measure (TCE) in the case of a multivariate gamma portfolio of risks. Explicit formulas for both the TCE and the risk capital allocations based on it are provided in the context of the multivariate model possessing dependent gamma marginals. Some of our results exceed the frameworks of the multivariate gamma distributions and may be applied to other non-negative risks.

Keywords: Tail conditional expectation; Multivariate gamma distribution

1. Introduction

In this paper, risks represent non-negative random variables such as they occur in the individual and collective models in actuarial risk theory. Consider a non-negative loss random variable X with density function $f_X(x)$, distribution function $F_X(x)$and tail function $\overline{F}_X(x) = 1 - F_X(x)$. The *tail conditional expectation* risk measure (TCE), also referred to as the *expected shortfall* (ES), is defined as

$$\mathrm{TCE}_X(x_q) = E(X|X > x_q). \tag{1}$$

Eq. (1) may be interpreted as the mean of the worst losses in the sense that it gives the average amount of the tail of the loss distribution. Such a tail is usually based on the qth quantile $x_q$of the latter. It is generally called "value-at-risk" and is denoted by $\mathrm{VaR}_X(q)$with

$$x_q = \inf\{x|F_X(x) \geq q\}, \tag{2}$$

where $0 < q < 1$.

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The formula used to evaluate tail conditional expectation is

$$\mathrm{TCE}_X(x_q) = \frac{1}{\bar{F}_X(x_q)} \int_{x_q}^{\infty} x \, \mathrm{d}F_X(x), \tag{3}$$

subject to $\bar{F}_X(x_q) > 0$.

Particular cases of TCE were studied by various authors. Thus, the expressions for the expected shortfall for the univariate and multivariate normal families were developed in Panjer and Jing (2001). Landsman and Valdez (2003) extended these results for the larger class of elliptical distributions. Unfortunately, all members of the elliptical family are symmetric, while insurance losses are in general modeled by non-negative and positively skewed random quantities.

Landsman and Valdez (2005) developed the TCE formulas for the univariate exponential dispersion family (EDF). The former includes many well-known distributions such as normal, gamma and inverse Gaussian, which, except for the normal, are not symmetric, have non-negative supports and provide an adequate model for fitting insurance losses. Although the univariate EDF is considerably rich and widely applied, the case is different for modeling the n-variate portfolios of financial risks and insurance claims distributed according to the multivariate EDF. The latter, for example, does not include important multivariate distributions whose univariate marginals are inverse Gaussian, gamma or stable (see Bildikar and Patil, 1968). Consequently, the multivariate EDF cannot be applied to model portfolios of risks with such claims.

Hürlimann (2001) seems to have been the first to consider TCE in the context of non-negative random variables. However, this author considered the expected shortfall in computing risk capitals for independent gamma sums, and the more general problem of managing a multivariate portfolio of dependent gamma risks was left untouched.

Gamma distributions have been widely applied in various fields of actuarial sciences. On one hand, these distribution functions possess non-negative supports and positive skewness, which are important for obtaining fair fitting results. On the other hand, gamma random variables have been well-studied, and they share many tractable mathematical properties which facilitate their use. There are numerous examples of applying Gamma approximations for modeling insurance portfolios (see Hürlimann, 2001; Melnick and Tenenbein, 2000; Rioux and Klugman, 2004). Gamma distributions provide a convenient model for the average rate of claims filed by different policyholders of an insurance company (see p. 152 of Herzog, 1996 or p. 98 of Hossack et al., 1983). Bowers et al. (1997) use a translated gamma distribution as a model for the aggregate claims of an insurance company.

In this paper, we base our risk analysis on the tail conditional expectation risk measure, since it shares a number of attractive properties. In particular, TCE allows for a natural allocation of the total loss among its various constituents. Assume that an insurance company manages n lines of business, and the risk managers are interested in quantifying the risk, which is concealed in line j. In the terms of TCE, the contribution of the jth line of business to the total risk capital of the whole insurance company is given by

$$\mathrm{TCE}_{X_j|S}(s_q) = E(X_j|S > s_q), \tag{4}$$

where $S = X_1 + X_2 + \cdots + X_n$ and s_q follows from (2). Certainly, due to the additive property of conditional expectations, the sum of all marginal risks is equal to the total risk measure, i.e.

$$\mathrm{TCE}_S(s_q) = \sum_{j=1}^{n} E(X_j|S > s_q). \tag{5}$$

We derive closed form expressions for (4) and (5) in the general frameworks of the multivariate model with univariate gamma marginals and the dependence structure based on the approach in Mathai and Moschopoulos (1991). Multivariate distributions with non-negative supports are in general very important in actuarial sciences, and therefore the suggested family may provide a good basis for modeling portfolios of insurance risks. The present paper is the first to consider a multivariate dependent non-negative setup as a TCE based risk capital allocation