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Fuzzy semi-Markov migration process in credit risk

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Abstract

We explore the rating system used by credit agencies with a focus on problems that justify the use of fuzzy set theory. We prove that a fuzzy market is viable if and only if an equivalent martingale measure exists, from which we construct the forward probability measure and under which the discounted price of a default-free bond is a martingale. We model the evolution of credit migration of a defaultable bond as an inhomogeneous semi-Markov process with fuzzy states. We study the effects of changing the real probability measure to a forward probability measure. In addition, we investigate the asymptotic behaviour of the survival probability in each fuzzy state given in the absence of default. Finally, we discuss parameter estimation and calibration of the inhomogeneous Markov chain with fuzzy states.

Keywords: Credit rating; Fuzzy markets; Inhomogeneous semi-Markov process with fuzzy states; Term structure in fuzzy markets

1. Introductory notes

Credit risk assessment depends on the use of a rating system. The credit quality of a corporate debt or a corporate bond is quantified, and categorized into a finite number of disjoint and mutually exclusive credit rating classes or credit grades. Corporate bonds are debt instruments issued by corporations, whereas state bonds are issued by states. We concentrate on discount bonds, that is, we assume that the bonds pay no coupons. We use the term *defaultable bond*, for any type of bond with a possibility of default. Most of the techniques presented here are applicable to the valuation of general corporate liabilities and corporate loans. However, merely for presentation purposes, we choose to limit the discussion to corporate bonds.

Credit rating classes are typically identified by elements of a finite set, referred to as the set of credit grades. The rating classes are usually attributed by a commercial rating agency, such as Moody's Investor service, Standard and Poor's, Fitch IBCA, and Duff and Phelps. A good account of the rating systems of these agencies could be found in [11]. Most rating systems involve both quantitative and qualitative evaluation. The final decision is based on many different attributes, but is typically not calculated using a formal model that would show how to weight these attributes in a normative way. In fact, the agency methodologies are non disclosed, which adds up to the vagueness of the situation. In essence, the rating systems are based on general considerations and on experience, rather than on mathematical modelling, which leads to a strong degree of subjectivity in the final decision. Hence, the rating classes cannot be regarded as precise and clearly rely on the judgement of the rating evaluators, which differ more often than they

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coincide. For the above reasons and others explained in the next section, we consider the credit rating classes of a system or firm as *fuzzy sets*.

After issuance and assignment of an initial bond rating, agencies review the underlying issues, although it is not clear if these are periodic or/and based on market events, probably both [1]. The stochastic process that represents the evolution over time of the rating grades of a defaultable bond is called the *migration process*. Fuzzy states for a stochastic process, especially those with a discrete state space, essentially occur in two types of situation [6]: (i) when the states of the system cannot be precisely measured, and thus the states used to model the system are intrinsically fuzzy and (ii) when the actual states can be exactly measured and are observable, but the number of states is so large that the decisions are not associated with the exact states of the system. In these situations, the decisions are associated with fuzzy states, which can be defined as fuzzy sets on the original non-fuzzy state space of the system.

The first attempt to model the evolution of credit migration was in terms of either a discrete or a continuous time homogeneous Markov chain (or conditionally homogeneous Markov chain) by Jarrow et al. [24,25]. Other researchers used a simple Markov chain approach [12,32–37]. Bielecki and Rutkowski reviewed the mathematical foundation of these models, their common core and interrelations [7]. Elliot et al. published a general paper on default risk [21] and Douady and Jeanblanc provided an extension to credit derivatives [18]. Using Moody's Investors Service database with data from 1976 to 1993, Carty and Fons found that the duration of a credit rating in a specific rating class followed a Weibull distribution [8]. The parameter estimates for the particular Weibull distribution varied for each credit rating class. Thus, the authors implicitly established that the appropriate model was not a simple Markov chain – in which case the duration of stay in each credit rating class would follow an exponential distribution (or the geometric distribution in the discrete case) – but rather was a semi-Markov process. Time inhomogeneity of transition probabilities has been reported by many authors [1,19,22,26,36,37]. According to these and other studies, real economic activity over the business cycle is correlated with fluctuations in transition and default probabilities.

Vasileiou and Vassiliou introduced a $\mathbb{G}$-inhomogeneous discrete time semi-Markov process to model the migration process [57]. The authors investigated the change from real-world probability to forward martingale measure and its influence on the inhomogeneous semi-Markov process and studied stochastic monotonicity problems. In addition, they proposed a term structure of credit spread and a model calibration. They also estimated real-world probabilities for the inhomogeneous semi-Markov process from typical data on credit migration. D'Amico et al. used the approach usually known as Markov renewal [13–16].

The remainder of the paper is organized as follows. Section 2 provides a brief overview of the rating system used by credit agencies. It also points out the problems that justify the use of fuzzy set theory. Membership functions as defined by Zadeh and their operations are also introduced [62]. In Section 3 we study a general discrete-time fuzzy market model, which consists of a savings account as a numèraire, a default-free zero-coupon bond and n defaultable bonds, where n is the number of fuzzy states distinguished for defaultable bonds. A series of theorems demonstrates that the fuzzy market is viable if and only if an equivalent martingale measure exists, from which we construct the forward probability measure and under which the discounted price of a default-free bond is a martingale.

In Section 4 we assume that the evolution of credit migrations of a defaultable bond is modelled as an inhomogeneous semi-Markov process with fuzzy states. Let $\{D_t\}_{t=0}^{\infty}$ be an inherent inhomogeneous Markov chain with state space $\mathbb{D}$ that models the evolution of the duration of the defaultable bond in each rating grade. Let $\{D_t^{(F)}\}_{t=0}^{\infty}$ with state space $\mathbb{D}^{(F)}$ be the corresponding inherent inhomogeneous Markov chain with fuzzy state space. We initially prove that under certain conditions easily met in practice, when the transition probability matrix among the non-default states of $\mathbb{D}$ is regular, then the transition probability matrix among the non-default states of the fuzzy state space $\mathbb{D}^{(F)}$ is also regular. In a series of theorems we study the asymptotic behaviour of the survival probabilities in each fuzzy state in the absence of default. In Section 5, we discuss the price of a defaultable bond in the fuzzy market. In particular, we prove that under certain conditions, changing the real probability measure to a forward probability measure does not affect the inherent inhomogeneous Markov chain with fuzzy state space, which still retains the Markov property. We also provide a relation among the transition probabilities under the two measures. In Section 6, we study parameter estimation and calibration of the inhomogeneous Markov chain with fuzzy state space.

2. Credit rating classes and fuzzy sets

We start with a brief overview of the rating system of credit agencies, with a focus on problems that justify the use of fuzzy set theory. Bond issuance by corporations is a 20th-century phenomenon. It started at the beginning of the