Journal of Banking & Finance 36 (2012) 1896–1910

Contents lists available at SciVerse ScienceDirect

Journal of Banking & Finance

journal homepage: www.elsevier.com/locate/jbf

Granularity adjustment for mark-to-market credit risk models

Michael B. Gordy [a,*], James Marrone [b]

[a] Federal Reserve Board, Washington, DC 20551, USA
[b] University of Chicago, Chicago, IL 60637, USA

ARTICLE INFO

Article history:
Received 17 June 2010
Accepted 28 February 2012
Available online 16 March 2012

JEL classification:
G32
G17

Keywords:
Granularity adjustment
Idiosyncratic risk
Portfolio credit risk
Value-at-risk
Expected shortfall

ABSTRACT

The impact of undiversified idiosyncratic risk on value-at-risk and expected shortfall can be approximated analytically via a methodology known as granularity adjustment (GA). In principle, the GA methodology can be applied to any risk-factor model of portfolio risk. Thus far, however, analytical results have been derived only for simple models of actuarial loss, i.e., credit loss due to default. We demonstrate that the GA is entirely tractable for single-factor versions of a large class of models that includes all the commonly used mark-to-market approaches. Our approach covers both finite ratings-based models and models with a continuum of obligor states. We apply our methodology to CreditMetrics and KMV Portfolio Manager, as these are benchmark models for the finite and continuous classes, respectively. Comparative statics of the GA reveal striking and counterintuitive patterns. We explain these relationships with a stylized model of portfolio risk.

Published by Elsevier B.V.

1. Introduction

In the portfolio risk-factor frameworks that underpin both industry models of credit value-at-risk (VaR) and the Internal Ratings-Based (IRB) risk weights of Basel II, credit risk in a portfolio arises from two sources, systematic and idiosyncratic. Systematic factors represent the effect of unexpected changes in macroeconomic and financial market conditions on the performance of borrowers. Borrowers may differ in their degree of sensitivity to systematic risk, but few firms are completely insulated from the wider economic conditions in which they operate. Therefore, the systematic component of portfolio risk is unavoidable and only partly diversifiable. Idiosyncratic factors represent the risks that are particular to individual borrowers. As a portfolio becomes more fine-grained, in the sense that the largest individual exposures account for a vanishing share of total portfolio exposure, idiosyncratic risk is diversified away at the portfolio level.

In some settings, including the IRB approach of Basel II, the computation of VaR is dramatically simplified under the assumption that bank portfolios are *perfectly* fine-grained, that is, that diversification fully eliminates idiosyncratic risk, so that portfolio loss depends only on systematic risk. Real-world portfolios are not, of course, perfectly fine-grained. When there are material name concentrations of exposure, there will be a residual of undiversified idiosyncratic risk in the portfolio. The impact of undiversified idiosyncratic risk on VaR can be approximated analytically via a methodology known as *granularity adjustment*. In principle, the granularity adjustment (GA) can be applied to any risk-factor model of portfolio credit risk. Thus far, however, analytical results have been derived only for simple models of *actuarial* loss, i.e., credit loss due to default. The implicit view appears to be that the GA would be tedious to derive, or perhaps even intractable, for the more complicated models of *mark-to-market* (MtM) credit loss. Large banks typically model credit loss in market value terms, and even the model underpinning the IRB approach of Basel II is in this advanced class.[1] In this paper, we demonstrate that the GA is in fact entirely tractable for a large class of models that includes single-factor versions of all the commonly used MtM approaches. If notation is chosen judiciously, the resulting derivations and calculations are concise and straightforward.

In Section 2, we review the established results in the literature on granularity adjustment and introduce the basic notation. Our general solution for mark-to-market models is given in Section 3. This solution covers both finite ratings-based models and models with a continuum of obligor states. In Section 4, we apply our methodology to CreditMetrics and KMV Portfolio Manager as these are the benchmark models for the finite and continuous classes,

* Corresponding author. Tel.: +1 202 452 3705.
E-mail addresses: michael.gordy@frb.gov (M.B. Gordy), jmar@uchicago.edu (J. Marrone).

[1] The IRB risk-weight formulae for corporate loans (Basel Committee on Bank Supervision, 2006, para 272) are organized in a way that visually suggests actuarial concepts, but the maturity adjustment maps to capital charges derived in a mark-to-market setting (see Gordy and Lütkebohmert, 2010, Section 1).

0378-4266/$ - see front matter Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.jbankfin.2012.02.010

respectively. In Section 5 we explore comparative statics. Some of the comparative statics appear counterintuitive at first glance, so in Section 6 we explain these results with a stylized MtM model of portfolio risk.

2. Granularity adjustment

For clarity in exposition, we first consider risk-measurement for a portfolio of n homogeneous positions. We wish to model the portfolio loss rate, $\widetilde{L}$, at a fixed horizon $t = H$ with current time normalized to $t = 0$. Let L_i denote the loss at the horizon on position i (expressed as a percentage of current value), so that the portfolio loss rate is simply

$$\widetilde{L} = \frac{1}{n}\sum_{i=1}^{n} L_i. \tag{1}$$

For a given target solvency probability $q \in (0,1)$, value-at-risk is the qth percentile of the distribution of loss. Let $\alpha_q(Y)$ denote the qth percentile of the distribution of random variable Y, i.e.,

$$\alpha_q(Y) \equiv \inf\{y : \Pr(Y \leqslant y) \geqslant q\}. \tag{2}$$

In terms of this more general notation, VaR is $\alpha_q(\widetilde{L})$.

Let X denote the set of systematic risk factors as realized at the horizon. A critical assumption of all risk-factor portfolio models is that all dependence in loss across positions is due to common dependence on X, so that L_i is independent of L_j when conditioned on X. As n grows to infinity, all idiosyncratic sources of risk vanish, so $|\widetilde{L} - E[\widetilde{L}|X]| \to 0$, almost surely. This implies that $\alpha_q(\widetilde{L}) \to \alpha_q(E[\widetilde{L}|X])$ as $n \to \infty$. This result is especially useful when X is univariate and conditional expected loss is increasing in X, and we henceforth impose these assumptions. Subject to mild restrictions, $\alpha_q(E[\widetilde{L}|X])$ is equal to $E[\widetilde{L}|X = \alpha_q(X)]$, which is easily calculated in analytical form.

The difference $\alpha_q(\widetilde{L}) - \alpha_q(E[\widetilde{L}|X])$ represents the effect of undiversified idiosyncratic risk in the portfolio. This difference is unobtainable in analytical form, but we construct an asymptotic approximation in orders of $1/n$,

$$\alpha_q(\widetilde{L}) - \alpha_q(E[\widetilde{L}|X]) = -\frac{1}{n}\frac{1}{2h(\alpha_q(X))} \times \frac{d}{dx}\left(\frac{V[L_1|X=x]h(x)}{\frac{dE[L_1|X=x]}{dx}}\right)\Bigg|_{x=\alpha_q(X)} + o(1/n), \tag{3}$$

where $h(\cdot)$ is the density of X and $V[Y]$ is the variance of random variable Y. The dominant term on the right hand side is the granularity adjustment.

The GA extends naturally to heterogeneous portfolios. Let A_i be the current size of exposure i. This is the face value of the instrument in an actuarial setting, and is the current market value in a mark-to-market setting. Let $a_i = A_i/\sum_{j=1}^{n} A_j$ be the portfolio weights. Imposing mild restrictions on the sequence $A_1, A_2, \ldots$ so that the $\sum_{i=1}^{n} a_i^2 \to 0$ as $n \to \infty$ (see Assumption $(\mathcal{A}-2)$ in Gordy, 2003), we have

$$\mathrm{GA} = \frac{-1}{2h(\alpha_q(X))}\frac{d}{dx}\left(\frac{V[\widetilde{L}|X=x]h(x)}{\frac{dE[\widetilde{L}|X=x]}{dx}}\right)\Bigg|_{x=\alpha_q(X)}. \tag{4}$$

In the risk management literature, Wilde (2001) is first to suggest this form of the GA. Martin and Wilde (2002) give a more rigorous derivation of Wilde's formula based on theoretical work by Gouriéroux et al. (2000). Gordy (2004) presents a survey of these developments and a primer on the mathematical derivation.[2]

The GA of Eq. (4) applies under either accounting paradigm for loss.[3] Under an actuarial definition, loss L_i on position i is the product of a default indicator for i and the loss given default (LGD) suffered on that position. LGD is expressed as a percentage of exposure and may itself be stochastic. Heretofore, all applications of the GA to portfolio credit risk have been in an actuarial setting. Wilde (2001) provides analytical solutions to Eq. (4) for the CreditRisk+ model and for an actuarial version of the CreditMetrics model. Emmer and Tasche (2005) develop analysis of CreditMetrics further. Even for the simple case of a homogeneous portfolio and zero recovery on defaulted loans, the Emmer and Tasche solution suggests some complexity. Expressed in the notation to be introduced below, we have

$$\begin{aligned}\mathrm{GA} = -\frac{1}{n}\frac{1}{2\frac{\sqrt{\rho}}{\sqrt{1-\rho}}\phi\left(\frac{C_0-\alpha_{1-q}(X)\sqrt{\rho}}{\sqrt{1-\rho}}\right)}&\left[\frac{\sqrt{\rho}}{\sqrt{1-\rho}}\phi\left(\frac{C_0-\alpha_{1-q}(X)\sqrt{\rho}}{\sqrt{1-\rho}}\right)\right.\\ &\times\left(1-2\Phi\left(\frac{C_0-\alpha_{1-q}(X)\sqrt{\rho}}{\sqrt{1-\rho}}\right)\right)\\ &+\left(\alpha_{1-q}(X)+\frac{\sqrt{\rho}}{\sqrt{1-\rho}}\frac{C_0-\alpha_{1-q}(X)\sqrt{\rho}}{\sqrt{1-\rho}}\right)\Phi\left(\frac{C_0-\alpha_{1-q}(X)\sqrt{\rho}}{\sqrt{1-\rho}}\right)\\ &\left.\times\Phi\left(\frac{\alpha_{1-q}(X)\sqrt{\rho}-C_0}{\sqrt{1-\rho}}\right)\right]\end{aligned} \tag{5}$$

where ϕ and Φ denote the standard normal density and cumulative probability functions. The original result, in (Emmer and Tasche (2005, Remark 2.3)), incorrectly has a minus sign in place of the second plus sign on the third line of Eq. (5). The same sign error occurs in the more general result in Proposition 2.2 of that paper. The obscurity of this error, which we believe to be unnoticed until now, perhaps reflects the opacity of the formulae.

In a mark-to-market setting, "loss" is an ambiguous concept. One needs to choose a reference point (i.e., the value of the instrument that counts as zero loss) and a convention for discounting to the present. A typical definition is the difference between expected return and realized return, discounted back to today at the riskfree rate.[4] Return is defined as the ratio of market value at the horizon (inclusive of cashflows received during the period $(0,H]$, accrued to the horizon at the riskfree rate) to the current market value. We adopt this convention, but note that it is generally trivial to modify our results to accommodate other definitions.[5]

To formalize, let $B_t(T)$ be the money market fund, i.e., $B_t(T)$ is the value at T of a unit of currency invested at date t in a riskless continuously compounded money market fund. We write this as

$$B_t(T) = \exp\left(\int_t^T r_s ds\right)$$

where r_t is the instantaneous short rate. Portfolio credit risk models generally exclude interest rate risk, so we assume that the path of r_t is deterministic (though not necessarily constant). We multiply intra-horizon cashflows by $B_t(H)$ to accrue to the horizon, and divide by $B_0(H)$ to discount horizon values back to today. Let W_i be the return on position i at the horizon, and define loss as $L_i = (E[W_i] - W_i)/B_0(H)$. Aggregate portfolio return is

[2] Voropaev (2011) offers an elegant alternative derivation. Gordy and Lütkebohmert (2010) address practical considerations for application to Basel II. Granularity adjustment also has applications in option pricing (Gagliardini and Gouriéroux, 2011), pricing and risk-measurement of CDOs (Antonov et al., 2007), econometrics (Gouriéroux and Monfort, 2009; Gouriéroux and Jasiak, 2012), simulation methods (Gordy and Juneja, 2010), and modeling systemic risk contributions in banking systems (Tarashev et al., 2010).

[3] When applied in a mark-to-market setting, mild additional restrictions are required to bound the conditional second moment of portfolio loss (Gordy, 2003, Section 3).

[4] The definition of loss is rarely made explicit in documentation of practitioner models. Some risk-management consultants use our definition. In CreditMetrics and KMV Portfolio Manager (as of version 1.4), loss is defined as the difference between expected value and realized value in time-H dollars, and it is left to the user to discount VaR to obtain a time-0 economic capital requirement. We divide by exposure size A_i to express loss in percentage terms.

[5] The GA is invariant with respect to the reference point for which "loss" is zero. Changing the discounting convention implies a linear rescaling of the GA.