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Stochastic optimal growth with risky labor supply[☆]

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ABSTRACT

Production takes time, and labor supply and profit maximization decisions that relate to current production are typically made before all shocks affecting that production have been realized. In this paper we re-examine the problem of stochastic optimal growth with aggregate risk where the timing of the model conforms to this information structure. We provide a set of conditions under which the economy has a unique, nontrivial and stable stationary distribution. In addition, we verify key optimality properties in the presence of unbounded shocks and rewards, and provide the sample path laws necessary for consistent estimation and simulation.

1. Introduction

In their 1983 paper "Real Business Cycles", John Long and Charles Plosser not only coined a phrase that every economist would soon come to know, they also helped lay the foundations of a new approach to modeling the business cycle. The particular version of the stochastic optimal growth model used by Long and Plosser in their study contains a notable feature: decisions regarding labor input must be made before current production shocks are realized. Subsequent research has carried on their agenda, but mainly using a different timing, where those making labor input decisions are permitted to observe the realized value of all shocks that will affect current production before choosing labor input (see Stokey et al., 1989 for an introductory treatment).

In empirical research, however, these production shocks are never directly observable; they are typically computed as residuals given data on output, capital, and labor. The now conventional shock–labor–output timing (the second timing discussed above) assumes that decision makers can observe these residuals before making labor input decisions. On the other hand, the labor–shock–output timing adopted by Long and Plosser is consistent with the view that decision makers have no more information than macroeconomists, and can observe (or calculate) the residuals only after observing output. Although the relative suitability of the two approaches will vary across different modeling applications, it seems hard to deny that the Long–Plosser approach has not received due attention in the literature.[1]

In this paper, our aim is to address fundamental aspects of the stochastic optimal growth model with Long and Plosser's timing, and provide the underlying results necessary for further research. We provide a detailed analysis of optimality and dynamics with general functional forms. (Long and Plosser's model specialized to the case of log utility and Cobb–Douglas production, which results in a linear law of motion for log output. This case is a useful benchmark, but is limited in the dynamics it is able to represent.) Our first significant contribution in this paper is to provide conditions under which a nontrivial stationary distribution for output exists, and for when it is unique and globally stable. These results are valuable because the dynamics of the stochastic optimal growth model with elastic labor and general functional forms are still largely unknown, both for the traditional timing *and* the timing studied here. This is remarkable, given that the dynamics in the inelastic labor

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[1] A number of papers working with the Long–Plosser timing can be found in the literature. For a recent example see Balbus et al. (2012, p. 8). What is lacking, however, is a foundational treatment like the one given in this paper.

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case were established so many years ago (Brock and Mirman, 1972; Mirman, 1973), that all realistic applications of this class of models allow labor to be endogenously supplied, and that almost all estimation and calibration techniques depend at a fundamental level on the existence of unique, nontrivial stationary solutions.[2]

We also establish geometric ergodicity of the output process, and laws of large numbers and central limit theorems for functions of output, investment and labor. These properties are fundamental to almost all quantitative analysis, and the sample path limit theorems are essential for simulation and estimation strategies. Finally, as an additional contribution, we provide weak conditions on the primitives of the model under which the Bellman equation holds, optimal policies exist and are unique, continuous and (in the case of savings and consumption) monotone. These conditions permit shocks, utility and the state space to be unbounded. Our approach to the dynamic programming problem is based on the use of weighted-supremum norms.

Regarding these results, it should be noted that the Long–Plosser timing used in this paper has some technical advantages vis-a-vis the standard timing, particularly when proving uniqueness and stability results for stationary equilibria. For example, the different timing leads to different state variables, and this difference between state variables means that, in the Long–Plosser timing, the next-period production shock appears outside rather than inside the optimal policy function. This makes it more feasible to assess the impact of these shocks. Nonetheless, our paper should provide a useful starting point for proving analogous results under the standard timing.

1.1. Related literature

The stochastic optimal growth model analyzed by Brock and Mirman (1972, 1973) motivated many subsequent studies aimed at characterizing optimal investment. See, for example, Mirman and Zilcha (1975), Razin and Yahav (1979), Donaldson and Mehra (1983), Brock and Majumdar (1978), Stokey et al. (1989), Hopenhayn and Prescott (1992), Mirman (1973), Stachurski (2002), Zhang (2007), Nishimura and Stachurski (2005), and Kamihigashi (2007). In all of these papers, labor is assumed to be inelastically supplied.

The joint behavior of capital and labor in stochastic dynamic recursive economies with market distortions and externalities has been considered in Greenwood and Huffman (1995), Coleman (1997), and Datta et al. (2002) under a set of conditions related to monotonicity of the marginal utilities. Most recently, under more general setting, Bosi and Le Van (2011), and Goenka et al. (2012) have studied similar problems in deterministic Ramsey models with and without borrowing constraints, respectively. In these papers, the focus is on the existence of competitive equilibria, and the problem of stability is largely untreated.

In the dynamic stochastic general equilibrium (DSGE) literature, models usually are approximated using Taylor expansions or similar techniques (e.g. Kydland and Prescott, 1982, Hansen, 1985, and Galí, 2008, Chapter 2). With this approach, the co-movements of capital investment and labor supply around the steady states or balanced growth path can be studied. However, it is not in general true that stability of the linear approximation implies stability of the original model (see Stachurski, 2007). Furthermore, the higher order properties that are eliminated may be critical to understanding actual dynamics (see Durlauf and Quah, 1999).

1.2. Structure of the paper

The rest of the paper is structured as follows. Section 2 sets up the model and studies the social planner's problem. Section 3 gives conditions under which a nontrivial stationary distribution of output exist. Section 4 presents results on stability and uniqueness, and on sample path properties such as the law of large numbers and the central limit theorem. Section 5 concludes. All proofs are deferred to the appendices.

2. The model

In this section, we first define the model and solve the social planner's problem. Below we let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_{++} := (0, \infty)$. For a generic function h, the symbols h'_i, h''_i and h''_{ij} refer to the first-order, second-order, and cross partial derivatives respectively, with i, j indexing the arguments.

2.1. Model and assumptions

We begin with an elementary description of the basic model suitable for optimization by a social planner. (There are no externalities or distortions in the model, and a discussion of decentralization can be found in Long and Plosser, 1983.) Final output is denoted y_t, and is treated as a state variable. It is observed at the start of period t and can be transformed one-for-one into current physical investment k_t. Investment and labor ℓ_t are choice variables, selected at the start of time t. A shock z_{t+1} is then revealed and production takes place, yielding at the start of next period

$$y_{t+1} = z_{t+1} F(k_t, \ell_t). \tag{1}$$

The convention with subscripts is that a time t subscript indicates that the variable lies in the time t information set and not the $t-1$ information set. In particular, z_{t+1} is not previsible at t. The function F represents the common production technology, and the shock z_{t+1} is aggregate. The value y_{t+1} that we refer to as "output" is more correctly thought of as the sum of current output and capital net of depreciation.[3]

The information structure adopted in (1) differs from the conventional specification $y_t = z_t F(k_t, \ell_t)$, under which the decision on labor input is made with the knowledge of the productivity shock that affects current production. This is because we follow Long and Plosser in assuming that labor supply is "risky"; in other words, the planner does not know the productivity shock when making decisions on labor input.

Our assumptions on the shock process are very standard:

Assumption 2.1. The shocks $\{z_t\}$ follow the Markov process on $\mathbb{R}_+$ given by

$$z_{t+1} = \psi(z_t, e_{t+1}), \qquad \{e_t\} \overset{\text{IID}}{\sim} \mu, \quad t = 0, 1, \ldots \tag{2}$$

The IID sequence $\{e_t\}$ is defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and takes values in a measurable space $(E, \mathscr{E})$ with common distribution μ. The function $\psi : \mathbb{R}_+ \times E \to \mathbb{R}_+$ is jointly measurable, and $z \mapsto \psi(z, e)$ is continuous and increasing on $\mathbb{R}_+$ for each $e \in E$.

We take Π to be the associated stochastic kernel (i.e., transition probability function), so that, in particular, $\Pi(z, B) = \mathbb{P}\{\psi(z, e_{t+1}) \in B\}$ for all $z \in \mathbb{R}_+$ and Borel subsets B of $\mathbb{R}_+$.[4]

2 That quantitative applications of the stochastic optimal growth model adopt endogenous labor supply is not surprising. Not only does endogenous labor supply add realism, it also permits modelers to address some of the most fundamental questions of macroeconomics. Fluctuations in employment and the co-movement of output, investment and labor supply are key phenomena of the business cycle. Because the efficiency of the labor market is a crucial determinant of the efficiency of the whole economy, labor–output dynamics have important implications for policy makers.

3 To be more explicit, we could take $F(k, \ell) := F_c(k, \ell) + (1-\delta)k$, where $F_c(k, \ell)$ is current output and δ parametrizes depreciation. It is easy to verify that if F_c satisfies the conditions of our assumptions below then so does F.

4 Some authors prefer to take the stochastic kernel Π as the primitive. The two approaches are equivalent, in the sense that every stochastic kernel on a completely