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RESEARCH PAPER

Risk Measure and Control Strategy of Investment Portfolio of Real Estate based on Dynamic CVaR Model

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Abstract: This article studies risk measure and control strategy of investment portfolio of real estate based on the dynamic condition value-at-risk (CVaR) model. A dynamic CVaR model is defined, which is a dynamic programming problem. It is shown that the dynamic CVaR problem is equal to another nonlinear programming problem. On the basis of dynamic CVaR model, a model of investment portfolio of real estate is built. The model is applied to compute investment proportion and risk losses of portfolio by using data of real estate of 10 cities in China. Numerical results show that the multistage investment has less risk of loss than the single-stage investment. The control strategy of risk is to choose investment proportion of portfolio according to low risk.

Key Words: dynamic conditional value-at-risk; CVaR; investment portfolio of real estate; risk measure

1 Introduction

In recent years, with the rapid development of real estate investment, the increasing cost of housing continuously leads to increase of investment in China. The higher house cost results in an increase in vacant houses. Meanwhile, low incomers do not have enough money to buy themselves houses. The investment in real estate has become an outstanding problem. The government in China has given a series of policies to control the investment in real estate such that it has well-balanced development.

The investment in real estate is a complex control problem, which has two kinds of risk: risk of system and risk of nonsystem. The risk of system cannot be dispersed by investment portfolio. But, the risk of nonsystem may be dispersed by investment portfolio so that the investor may obtain stable returns. In recent years, some theories of investment portfolio in finance have been applied to study investment in real estate[1−4]. Most of the studies use the famous Markowitz model of investment portfolio in one stage. In fact, investment in real estate is a dynamic multistage control problem, which suffers from many unstable factors. Therefore, it is necessary to apply a more complex model to study investment in real estate.

This article uses a new tool of risk measure, a dynamic CVaR model, to study investment in real estate. Since Rockafell and Uryasev proposed the CVaR model, theory and application of CVaR have rapidly developed[5]. Krokhmal, Palmquist, and Uryasev applied the CVaR model to study portfolio and showed that the CVaR model is efficient for computing hundreds or thousands of stocks as portfolio[6]. Meanwhile, CVaR has many good properties, such as being computable, convex, and so on. Especially, CVaR is more efficient than the Markowitz value-at-risk (VaR) model for portfolio. The CVaR model has been applied to many other fields, such as an electric energy market[7], credit markets[8], enterprise risks[9], water resources development[10], newsvendor problem[11], multiproduct ordering problem[12], pool-based electricity markets[13], and so on.

In this article, the dynamic CVaR model, which includes state vector, control vector, state transfer equation, α-VaR value, and α-CVaR value at every stage and all stages are defined. Then, a model of investment portfolio of real estate based on the dynamic CVaR model is introduced. By applying the model to sample data of housing cost in 10 cities in China, numerical results of CVaR value and investment portfolio are obtained. Numerical results show that risk of investment portfolio in multiple stages is lower and distinct. Investment in real estate in most cities of China has obvious risk, according to the computing results. The model provides an important method for investment portfolio of real estate.

2 Dynamic CVaR model

Now, it is considered that the real estate portfolio investment is a dynamic programming model. The authors shall prove that the model equals to solve another nonlinear programming problem.

Let a stage of investment be i. Let $f_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \boldsymbol{\xi}) : R^m \times R^n \times R^r \to R^1$ $(i = 1, 2, \cdots, I)$ be a loss function for a control vector $\boldsymbol{x}_i \in X_i \subset R^n$ and a state vector $\boldsymbol{s}_i \in \boldsymbol{S}_i \subset R^n$ with a random vector $\boldsymbol{\xi} \in R^r$ under a density function $p(\boldsymbol{\xi})$. Let $X_i(\boldsymbol{s}_i)$ $(i = 1, 2, \cdots, I)$ be a set of all control vectors under a state vector $\boldsymbol{s}_i$. Let a state transfer equation be denoted by

$$\boldsymbol{s}_{i+1} = g_i(\boldsymbol{s}_i, \boldsymbol{x}_i), \boldsymbol{s}_i \in \boldsymbol{S}_i, \boldsymbol{x}_i \in X_i(\boldsymbol{s}_i), i = 1, 2, \cdots, I-1, \tag{1}$$

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where $\boldsymbol{s}_1$ is a start point. $\boldsymbol{x}_i(\boldsymbol{s}_i) \in X_i(\boldsymbol{s}_i)$ is a control vector at stage i.

For $i = 1, 2, \cdots, I$, let a distribution function $\Psi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \cdot)$ of $f_i(s_i, \boldsymbol{x}_i, \boldsymbol{\xi})$ be defined by:

$$\Psi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y) = P\{f_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \boldsymbol{\xi}) \le y\} = \int\limits_{f_i(s_i, x_i, z) \le y} p(\boldsymbol{z})\mathrm{d}\boldsymbol{z}. \tag{2}$$

Let a state sequence $\boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_i)$ and a control sequence $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_i)$ combine a strategy $(\boldsymbol{s}, \boldsymbol{x})$. All strategies are defined by a set A: $A = \{(\boldsymbol{s}, \boldsymbol{x}) | \boldsymbol{s}_{i+1} = g_i(\boldsymbol{s}_i, \boldsymbol{x}_i), i = 1, 2, \cdots, I-1, \boldsymbol{s}_i \in \boldsymbol{S}_i, \boldsymbol{x}_i \in X_i(\boldsymbol{s}_i), i = 1, 2, \cdots, I.\}$

Definition 2.1 *Given a confidence level $\alpha \in (0, 1)$ and a strategy $(\boldsymbol{s}, \boldsymbol{x})$.*

$$y_\alpha^*(\boldsymbol{s}_i, \boldsymbol{x}_i) = \min\{y | \Psi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y) \ge \alpha\} \tag{3}$$

is called an α-VaR loss value with confidence level α to $\boldsymbol{x}_i$ and $\boldsymbol{s}_i$.

Denote $\boldsymbol{y} = (y_1, y_2, \cdots, y_i)$. For each $i = 1, 2, \cdots, I$, let

$$\Phi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y) = (1-\alpha)^{-1} \int\limits_{f_i(s_i, x_i, z) \ge y} f_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \boldsymbol{z}) p(\boldsymbol{z})\mathrm{d}\boldsymbol{z}. \tag{4}$$

$$\Phi(\boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{I} \Phi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_i).$$

Definition 2.2 *$\Phi(\boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y}_\alpha^*(\boldsymbol{s}, \boldsymbol{x}))$ is called α-CvaR loss value with confidence level α to $(\boldsymbol{s}, \boldsymbol{x})$ at t, where*

$$\boldsymbol{y}_\alpha^*(\boldsymbol{s}, \boldsymbol{x}) = (y_\alpha^*(\boldsymbol{s}_1, \boldsymbol{x}_1), y_\alpha^*(\boldsymbol{s}_2, \boldsymbol{x}_2), \cdots, y_\alpha^*(\boldsymbol{s}_I, \boldsymbol{x}_I)).$$

For $i = 1, 2, \cdots, I$, a loss function is defined by

$$F_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y) = y + (1-\alpha)^{-1} \int\limits_{z \in R^r} [f_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \boldsymbol{z}) - y]^+ p(\boldsymbol{z})\mathrm{d}\boldsymbol{z}. \tag{5}$$

According to reference [5], the following result is obtained.

Lemma 2.1 *For $i = 1, 2, \cdots, I$, if*

$$P\{f_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \boldsymbol{\xi}) = \bar{y}\} = \int\limits_{f_i(s_i, x_i, z) = \bar{y}} p(\boldsymbol{z})\mathrm{d}\boldsymbol{z} = 0,$$

Then $F_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y)$ is continuous and convex to y, and

$$\min_{y \in R} F_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y) = \Phi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_i^*(\boldsymbol{x}_i)),$$

$$\frac{\partial F_i(s_i, x_i, y)}{\partial y} = (1-\alpha)^{-1}[\Psi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y) - \alpha].$$

By Lemma 2.1, a method for finding out optimal α-CVaR value can be obtained. A strategy $(\boldsymbol{s}, \boldsymbol{x}) \in A$ has to be found in all stages such that $\Phi(\boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y}_\alpha^*(\boldsymbol{s}, \boldsymbol{x}))$ is minimum. Consider the following dynamic CVaR problem.

(CVaR) $\min \Phi(\boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y}_\alpha^*(\boldsymbol{s}, \boldsymbol{x})) = \sum\limits_{i=1}^{I} \Phi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_\alpha^*(\boldsymbol{s}_i, \boldsymbol{x}_i))$

s.t. $(\boldsymbol{s}, \boldsymbol{x}) \in A$.

At stage k, for each state $\boldsymbol{s}_k$,

$$\Phi_k(\boldsymbol{s}_k, \boldsymbol{x}_k) =: \sum_{i=k}^{I} \Phi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_\alpha^*(\boldsymbol{s}_i, \boldsymbol{x}_i)),$$

may be denoted where a state subsequence $\boldsymbol{s}_k = (\boldsymbol{s}_k, \boldsymbol{s}_{k+1}, \cdots, \boldsymbol{s}_I)$ with respective to a control subsequence $\boldsymbol{x}_k = (\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \cdots, \boldsymbol{x}_I)$. Then, $(\boldsymbol{s}_k, \boldsymbol{x}_k)$ is called a substrategy from stage k at state $\boldsymbol{s}_k$ to stage I. Set of all substrategies is denoted by $A_k(\boldsymbol{s}_k)$: $A_k(\boldsymbol{s}_k) = \{(\boldsymbol{s}_k, \boldsymbol{x}_k) | \boldsymbol{s}_{i+1} = g_i(\boldsymbol{s}_i, \boldsymbol{x}_i), i = k, k+1, \cdots, I-1, \boldsymbol{x}_i \in X_i(\boldsymbol{s}_i), i = k, k+1, \cdots, I.\}, \boldsymbol{s}_i \in \boldsymbol{S}_i$, So, k-subproblem of dynamic CVaR problem is defined as follows.

$$\Phi_k(\boldsymbol{s}_k) = \min\{\Phi_k(\boldsymbol{s}_k, \boldsymbol{x}_k) | (\boldsymbol{s}_k, \boldsymbol{x}_k) \in A_k(\boldsymbol{s}_k)\}.$$

According to the optimal principle of dynamic programming, a solution to (CVaR) can be obtained by solving all the following subproblems.

$$\Phi_i(\boldsymbol{s}_i) = \min_{x_i \in X_i(s_i)} \{\Phi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_\alpha^*(\boldsymbol{s}_i, \boldsymbol{x}_i)) + \Phi_{i+1}(g_i(\boldsymbol{s}_i, \boldsymbol{x}_i))\}, \quad i = 1, 2, \cdots, I,$$

$\Phi_{I+1}(\boldsymbol{s}_{i+1}) = 0.$

By (4), it is difficult to solve $\Phi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_\alpha^*(\boldsymbol{s}_i, \boldsymbol{x}_i))$. Hence, it is not easy to solve (CVaR) and all the above subproblems. Now, another problem may be solved as follows.

(FCVaR) $\min \sum\limits_{i=1}^{I} F_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_i)$

s.t. $\boldsymbol{y} \in R^I, (\boldsymbol{s}, \boldsymbol{x}) \in A$.

The solution to (CVaR) can be obtained by solving (FCVaR). The following theorem can be proved.

Theorem 2.1 *For given $(\boldsymbol{s}, \boldsymbol{x}) \in A$, if $\bar{y}$ is an optimal solution to $\min\limits_{y \in R^I} \sum\limits_{i=1}^{I} F_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_i)$, and*

$$P\{f_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \boldsymbol{\xi}) = \bar{y}_i\} \int\limits_{f_i(s_i, x_i, z) = \bar{y}_i} p(\boldsymbol{z})\mathrm{d}\boldsymbol{z} = 0, i = 1, 2, \cdots, I,$$

then

$$\sum_{i=1}^{I} \Phi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \bar{y}_i) = \sum_{i=1}^{I} F_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_i), \tag{6}$$

and $\bar{y}_i = y_\alpha^*(\boldsymbol{s}_i, \boldsymbol{x}_i), i = 1, 2, \cdots, I$. Furthermore, if $(\bar{s}, \bar{x}, \bar{y})$ is an optimal solution to (FCVaR), then $(\bar{s}, \bar{x})$ is an optimal solution to (CVaR).

Proof. Denote $F(\boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y}) = \sum\limits_{i=1}^{I} F_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_i)$. By Lemma 2.1, we know that $F(\boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y})$ is continuous and convex at $\boldsymbol{y}$ and for each I,

$$\frac{\partial F_(s, x, y)}{\partial y_i} = \frac{\partial F_i(s_i, x_i, y_i)}{\partial y_i} = (1-\alpha)^{-1}(\Psi_i(s_i, x_i, y_i) - \alpha). \tag{7}$$

As $\bar{y}$ is an optimal solution to $\min\limits_{y \in R^I} \sum\limits_{i=1}^{I} F_i(\boldsymbol{s}_i, \boldsymbol{x}_i, y_i)$, then above equation (7) is 0, that is,

$$\Psi_i(\boldsymbol{s}_i, \boldsymbol{x}_i, \bar{y}_i) - \alpha = 0, i = 1, 2, \cdots, I.$$