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Markowitz's mean–variance defined contribution pension fund management under inflation: A continuous-time model☆

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HIGHLIGHTS

- The mean–variance criterion is used in our DC pension optimal investment problem.
- Stochastic inflation is considered and the assets on the market can all be risky.
- The Lagrange dual theory, the HJB approach, and the PDE technique are used.
- Explicit expressions for the efficient strategy and efficient frontier are derived.
- Two degenerate cases are discussed, and several numerical examples are presented.

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ABSTRACT

In defined contribution (DC) pension schemes, the financial risk borne by the member occurs during the accumulation phase. To build up sufficient funds for retirement, scheme members invest their wealth in a portfolio of assets. This paper considers an optimal investment problem of a scheme member facing stochastic inflation under the Markowitz mean–variance criterion. Besides, we consider a more general market with multiple assets that can all be risky. By applying the Lagrange method and stochastic dynamic programming techniques, we derive the associated Hamilton–Jacobi–Bellman (HJB) equation, which can be converted into six correlated but relatively simple partial differential equations (PDEs). The explicit solutions for these six PDEs are derived by using the homogenization approach and the variable transformation technique. Then the closed-form expressions for the optimal strategy and the efficient frontier can be obtained through the Lagrange dual theory. In addition, we illustrate the results by some numerical examples.

1. Introduction

Pension funds may be classified as defined benefit (DB) or defined contribution (DC), according to how the benefits are determined (see Xiao et al., 2007). A DB pension fund is a pension scheme in which the benefits are fixed in advance by the sponsor, and contributions are set and subsequently adjusted so as to ensure that the fund remains in balance; hence the risk is borne by the sponsor. A DC pension fund is a pension scheme in which only the contributions are fixed, and therefore the benefits depend solely on the returns of the fund's portfolio; hence the financial risk is borne by the contributors (see Boulier et al., 2001).

For most of the 20th century, employer-sponsored DB pension funds were the type preferred by many workers, since the risk is borne by the sponsors. However, with high inflation and an aging population, the total value of DB pension funds has been decreasing unceasingly, and the funds face great pressure to pay enough in annuities to the retirees. In recent years, most of the pension funds have been based on DC, such as Individual Retirement Accounts in the USA and Appropriate Personal Pension in the UK.

Nowadays, DC pension funds are playing an increasingly important role in social security systems all over the world. DC pension fund management has also become a popular topic over the last decade in the literature of actuarial and financial studies. We list some in the following.

By minimizing the risk that is a quadratic target-based (mean-squared error) cost function, Vigna and Haberman (2001) and

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Haberman and Vigna (2002) investigate a discrete-time DC pension fund management problem during the accumulation phase. The optimal investment strategies are obtained by using the dynamic programming method. Gerrard et al. (2004) extend the work of Vigna and Haberman (2001) to a continuous-time setting and the case of post-retirement. By maximizing the expected utility of the fund wealth in excess of the minimum guarantee at the retirement date, Boulier et al. (2001) and Deelstra et al. (2003) consider an optimal asset allocation problem with stochastic interest rates and a minimum guarantee protection. Deelstra et al. (2004) further study the optimal design of the minimum guarantee by using the martingale method. Xiao et al. (2007), Gao (2009, 2010) investigate the optimal portfolios under a constant elasticity of variance (CEV) model by applying the Legendre transform and the dual theory. Korn et al. (2011) consider a portfolio selection problem in a hidden Markov regime switching market. Using the method of stochastic control and optimal stopping, Gerrard et al. (2012) formulate and solve the problem of optimal annualization time for retirees in the decumulation phase. For more detailed discussion on DC pension fund management, we refer to Cairns et al. (2006), Giacinto et al. (2011), Giacinto and Vigna (2012), Emms (2012), and Blake et al. (2013).

In pension fund management, one should consider broad categories of risks such as investment or market risk, counter-party default risk, solvency risk, and liquidity risk. For long-term pension fund managers, since the accumulated inflation can lead to huge shrinkage in the wealth of the fund, inflation risk is an important factor that can affect the overall performance of the fund. General portfolio selection problems incorporating the inflation risk for individual investors have been considered in some papers; see Brennan and Xia (2002) and Munk et al. (2004). Recently, the optimal investment problem in DC pension fund management has become popular in the literature. To maximize the exponential utility of the fund wealth in a continuous-time setting, Battocchio and Menoncin (2004) and Ma (2011) study an optimal asset allocation problem with inflation. Zhang and Ewald (2010) and Han and Hung (2012) solve an expected power utility maximization problem with inflation by using the martingale approach and the method of stochastic control, respectively.

However, all the above-mentioned literature on DC pension fund management with inflation is based on expected utility maximization. Along another line, Markowitz (1952) sets up the famous mean–variance (M–V) model, which laid the foundation for modern portfolio selection theory. Since in practice the investment decision is a long-term dynamic process, Li and Ng (2000) and Zhou and Li (2000) extend the Markowitz M–V model to a multi-period case and a continuous-time case, respectively. In recent years, the M–V criterion has been studied by many scholars in various areas, for example, optimal investment and/or reinsurance (see Delong and Gerrard, 2007, Bai and Zhang, 2008 and Li et al., 2012) and asset–liability management (see Chiu and Li, 2006, Chen et al., 2008, Leippold et al., 2011 and Yao et al., 2013a,b).

As far as we know, very few investigations apply the M–V criterion to study the investment problem in the management of pension funds. In a continuous-time setting, Delong et al. (2008) and Josa-Fombellida and Rincón-Zapatero (2008) study the optimal investment and contribution strategies for DB pension funds; Vigna (2012) investigates the optimal investment strategy for DC pension funds. However, they do not incorporate the risk of inflation. In this paper, we are going to study an optimal investment problem with stochastic inflation in DC pension fund management under the M–V criterion. Besides, since the DC pension fund investment problem usually considers a long time horizon, say, 20–40 years, a risk-free asset hardly exists in the financial market. Hence, in our portfolio selection panel, we consider a more general model in which all assets could be risky.

The remainder of this paper is organized as follows. Section 2 provides the formulation of the problem. In Section 3, the associated nonlinear Hamilton–Jacobi–Bellman (HJB) equation is obtained and converted into six relatively simple partial differential equations (PDEs). Section 4 gives the integral-form analytical solutions for these six PDEs. The numerical calculation method for the corresponding integrals is also discussed in this section. The closed-form expressions for the efficient strategy and the M–V efficient frontier are obtained in Section 5. Section 6 gives some numerical results, and we conclude in Section 7.

2. Model formulation

We consider a financial market consisting of $n+1$ risky assets with prices at time t denoted by $P_0(t), P_1(t), \ldots, P_n(t)$. Let A' be the transpose of a matrix or vector A. Throughout the paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space on which we define an m-dimensional standard Brownian motion $w(t) = (w_1(t), w_2(t), \ldots, w_m(t))'$; hence $\mathcal{F} = \{\mathscr{F}_t; t \geqslant 0\}$, where $\mathscr{F}_t = \sigma\{w(s); 0 \leqslant s \leqslant t\}$ is generated by the Brownian motion and $\mathbb{P}$ represents the probability measure. The filtration $\mathscr{F}_t$ can be interpreted as the information set available up to time t. Assume that all the prices of these risky assets are modeled by geometric Brownian motions, i.e.,

$$\begin{cases} dP_i(t) = P_i(t)\left(b_i(t)dt + \displaystyle\sum_{j=1}^{m} \sigma_{ij}(t)dw_j(t)\right), \\ P_i(0) = p_i^0, \quad i = 0, 1, 2, \ldots, n, \end{cases} \tag{1}$$

where $b_i(t)$ and $\sigma_i(t) = (\sigma_{i1}(t), \sigma_{i2}(t), \ldots, \sigma_{im}(t))$ are the appreciation rate and the volatility rate for the ith risky asset, respectively.

Remark 1. Here, we include the case with a risk-free asset. In fact, let $b_0(t) = r_f(t)$ and $\sigma_{0j}(t) = 0$ for $j = 1, 2, \ldots, m$, where $r_f(t)$ denotes the instantaneous risk-free interest rate. Then the 0th asset can be taken as a risk-free asset.

As adopted in much literature (see Brennan and Xia, 2002 and Munk et al., 2004), we suppose that the price index $\Pi(t)$ satisfies the following system of stochastic differential equations:

$$\begin{cases} d\Pi(t) = \Pi(t)\{I(t)dt + \sigma_\Pi(t)dw(t)\}, \\ \Pi(0) = \Pi_0, \end{cases} \tag{2}$$

where $\sigma_\Pi(t) = (\sigma_{\Pi 1}(t), \sigma_{\Pi 2}(t), \ldots, \sigma_{\Pi m}(t))$ is the volatility of the price index, and $I(t)$ is the instantaneous expected inflation rate following an Ornstein–Uhlenbeck process

$$\begin{cases} dI(t) = \psi(t)(\bar{I}(t) - I(t))dt + \sigma_I(t)dw(t), \\ I(0) = I_0, \end{cases} \tag{3}$$

where $\bar{I}(t)$ is the long-run mean of the inflation rate, $\psi(t)$ is the degree of mean reversion, and $\sigma_I(t) = (\sigma_{I1}(t), \sigma_{I2}(t), \ldots, \sigma_{Im}(t))$ is the volatility of the inflation rate.

It should be noted that the asset prices, the price index, and the expected inflation rate could be correlated with each other, since they are all affected by the common stochastic factor $w(t)$.

In this paper, we investigate a portfolio selection problem in the accumulation phase of a DC pension fund. Consider a pension fund member who enters the plan at time 0 with an initial fund x_0 ($x_0 \geqslant 0$) in his/her account. He/she contributes the account continuously in a predefined way until his/her retirement at time T. Upon his/her retirement, he/she can withdraw all the money or convert it into an annuity. Let $C(t)$ denote the accumulated value of the contribution at time t. For simplicity, and without loss of generality, the dynamics of $C(t)$ can be modeled as follows:

$$dC(t) = \Pi(t)c(t)dt, \tag{4}$$