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Reactive investment strategies

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ABSTRACT

Asset liability management is a key aspect of the operation of all financial institutions. In this endeavor, asset allocation is considered the most important element of investment management. Asset allocation strategies may be static, and as such are usually assessed under asset models of various degrees of complexity and sophistication. In recent years attention has turned to dynamic strategies, which promise to control risk more effectively.

In this paper we present a new class of dynamic asset strategy, which respond to actual events. Hence they are referred to as 'reactive' strategies. They cannot be characterized as a series of specific asset allocations over time, but comprise rules for determining such allocations as the world evolves. Though they depend on how asset returns and other financial variables are modeled, they are otherwise objective in nature.

The resulting strategies are optimal, in the sense that they can be shown to outperform all other strategies of their type when no asset allocation constraints are imposed. Where such constraints are imposed, the strategies may be demonstrated to be almost optimal, and dramatically more effective than static strategies.

1. Introduction

Asset allocation strategies are generally of three types:

(a) Fixed asset allocations through time (typically 70% equity and 30% debt for long term investors, such as pension funds). This allocation is reviewed infrequently.
(b) Asset allocations that evolve over time according to a fixed and pre-determined schedule. Examples are so-called 'lifecycle' funds for superannuation investors. These have a fixed horizon and generally become more defensive as an investor approaches retirement.
(c) Rebalancing *rules*, under which the allocation at any future time is not pre-determined, but varies according to the actual investment experience up to that time.

Strategies (a) and (b) have been the most commonly used in practice. However in recent years, attention has focused on (c), which are potentially effective in controlling risk.

Generally an investment portfolio can drift from its target asset allocation, acquiring risk and return characteristics that may be inconsistent with an investor's goals and preferences. A rebalancing strategy addresses this situation by formalizing guidelines about how frequently the portfolio should be monitored, how far an asset allocation can deviate from its target before it is rebalanced, and whether periodic rebalancing should restore a portfolio to its target or to some other allocation. The question is: 'is there is an optimal rebalancing strategy in any sense?'. This is clearly a multi-period issue, formulated in either discrete or continuous time.

There is a wealth of literature on how risk factors and asset classes can be modeled under approach (a). We mention only a few of them here. Martinelli (2006) provides a formal but simplified model of assets and liabilities, identifying factors which are common to them both. Using stochastic optimization, this framework leads to appropriate asset allocations which support the investor's funding objectives.

Dempster et al. (2009) adopt an econometric model covering a variety of economic, financial and asset return variables, with complex interactions between them. These are more than adequate in capturing the effect of the risk factors that drive most liabilities in practice. When coupled with investor objectives as to funding levels, an optimal asset allocation strategy may be formulated by dynamic stochastic programming, i.e. by optimization over simulated outcomes.

Dhaene et al. (2005) avoid the use of simulation by approximating the copula of the dependent variables with a comonotonic copula. They do so under a lognormal framework for asset returns, employing a comonotonic copula to derive upper and lower bounds for the distribution of accumulated outcomes. More recently,Van Weert et al. (2010) extends these results to situations where cash flows are unrestricted in sign, thereby allowing savings towards pension benefits to be investigated.

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More recently, Li et al. (2006) also employ optimization to derive a closed form solution for the constant rebalanced portfolio (CRP—i.e. a static asset allocation) where the risk measure is based on a tail-VaR.

One of the earliest implementations of type (b) was due to Cariño et al. (2006). This study is focused on maximizing terminal outcomes for an income-paying insurance product subject to a host of regulatory constraints. The solutions are reached via stochastic dynamic programming.

More theoretical approaches are employed in Gerber and Shiu (2000) to obtain exact solutions under a two-asset model, with various forms of the utility of terminal outcomes. Munk (2008) provides a comprehensive review of dynamic asset strategies under various types of asset model and investor objectives. Cairns et al. (2006) provide a lifetime approach to saving for providing retirement income. Bruynel (2005) applies these techniques to a variety of situations, using a simple Markovitz model for asset returns. Most recently Huang and Lee (2010) apply this technique to the management of a life insurance portfolio where asset returns are subject to an autoregressive process.

The asset strategies of type (b) are dynamic in the sense that they change over time. These naturally lead to 'lifecycle' or 'target date' savings products, where the asset allocation evolves over the lifetime of the investor. However they do so in a pre-determined way, and that might explain the limited popularity they have enjoyed in practice in some countries.

Basu and Drew (2009) contest the viability of lifecycle funds which become more defensive towards the end of the investor's horizon. They show empirically that this defensiveness is costly in terms of overall return, especially where funding is weighted towards the end of the investor's horizon. More recently they suggest a dynamic strategy (Basu et al., 2011), where the weighting to growth assets depends on the actual performance relative to a target. Their suggestion is thus for a strategy of type (c).

The alternative approach under (c) is responsive to how the asset strategy performs in practice. This may allay some concerns of the investor that he is locked into an inflexible, and possibly destructive, path.

Seth (2002) takes up the issue of dynamic rebalancing to a target asset allocation. As a result of transaction costs, it is more effective to establish an asset allocation *range* for rebalancing, rather than rebalancing precisely to the target. Tokat and Wicas (2007) suggest that the frequency of rebalancing should depend on the direction of markets as well as on the target allocation adopted.

Bone and Goddard (2009) take up this theme in the context of funding pension liabilities. A contrarian rule is one of weighting to growth assets when pension funding is poor, and vice versa. Under the contrarian rebalancing strategy, a target performance might be set as, say, 3% pa. above annual inflation. Any surplus relative to the target will be invested into bonds, while any deficiency will be invested into equities. This may be contrasted with the CRP approach, where a portfolio is rebalanced to a predetermined asset allocation. They demonstrate that both on a historical, as well as on a prospective, basis this type of strategy can be superior to a static CRP of type (a). It is certainly superior to one of a 'casino' type, where more is allocated to growth assets when pension funding is favorable.

The apparent success of type (c) strategies is intriguing. In this paper we demonstrate that this success is not accidental, under a variety of conditions. We specifically derive the optimality criterion, and illustrate its outcomes, even in the presence of complex regulatory constraints. This is provided in the generic context of an investor with deterministic cash flows, but may be generalized to institutional investors with uncertain liabilities. Nonetheless, the need for an appropriate asset model, and identification of appropriate investor objectives, remain fundamental to all these problems.

1.1. Background

An investor has at time t accumulated wealth A_t as a result of previous investment returns. If the return for the period $[t, t+1]$ is denoted r_t and the cash flow at the end of this period is C_{t+1}, then the accumulations are governed by the simple relationship

$$A_{t+1} = (1 + r_t) A_t + C_{t+1}. \tag{1}$$

Without loss of generality, we can take $A_0 = 1$. Note that an initial cash flow C_0 is then not needed.

As a simplifying assumption, the returns r_t are taken to be independent between periods and are derived from a portfolio with mean return μ_t and volatility $\sigma_t = \sigma(\mu_t)$. This assumption will be relaxed in later sections of this paper, but note that no assumptions are made about the shape of the return distribution. Here the function $\sigma(\cdot)$ describes the efficient frontier for a given set of m asset classes and asset assumptions in accordance with the conventional Markovitz framework.

Following Bruynel (2005), an *investment strategy* is defined as a sequence $M = \{\mu_t | t = 0, 1, \ldots, n-1\}$ of expected returns until some horizon $t = n$, which specify a set of portfolios. We allow $\mu_t = \mu_t(A_t)$ to depend only on accumulated wealth to date, so that the strategy is of type (c). This is reasonable, given that asset returns are independent between periods, and suggests that the strategy is path-independent in the sense of Cox and Leland (2000). Again this assumption is relaxed later in this paper.

The investor's optimization problem is therefore to optimize terminal outcomes for A_n by specifying the form of the expected returns $\mu_t(A_t)$. Here we take optimization to mean finding the minimum variance $\mathbf{var}(A_n)$, or equivalently $\mathbf{E}\left(A_n^2\right)$, for a required expectation of terminal wealth $\mathbf{E}(A_n)$. For convenience we denote the standard deviation of A_n as $\mathbf{sd}(A_n) = \sqrt{\mathbf{var}(A_n)}$.

Detailed proofs and other results are provided in the Appendix.

2. Efficient frontier

Suppose there are m asset classes, with an expected return vector $\boldsymbol{\gamma}$ and covariance matrix $\boldsymbol{\Sigma}$. Formally we adopt the following process for the returns $\boldsymbol{r}_t$:

$$\mathbf{r}_t = \boldsymbol{\gamma} + \mathbf{e}_t$$

where the innovations $\mathbf{e}_t$ are independent and identically distributed (iid) with covariance matrix $\boldsymbol{\Sigma}$. We do not need, however, to specify the precise distribution. As in practice we do not observe any asset that is perfectly riskless, we take $\boldsymbol{\Sigma}$ to be positive-definite.

For a weighting $\mathbf{w}$ to the asset classes, the portfolio return $\mu = \boldsymbol{\gamma}^T\mathbf{w}$ and portfolio variance is $\sigma^2 = \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$. In practice the weights $\mathbf{w}$ are subject to linear constraints of the form $\mathbf{a}_i^T\mathbf{w} \le b_i$. We then have the following result, which is proved in the Appendix.

Proposition 1. *Suppose the constraints $\mathbf{A}^T\mathbf{w} = \mathbf{b}$ are binding on the portfolio weights $\mathbf{w}$ for some region of the efficient frontier. Then the efficient frontier $\sigma = \sigma(\mu)$ is quadratic, with the general form*

$$\sigma^2 = \begin{bmatrix} \mu \\ 1 \\ \mathbf{b} \end{bmatrix}^T \mathbf{Q} \begin{bmatrix} \mu \\ 1 \\ \mathbf{b} \end{bmatrix}$$

for $\mathbf{Q} = \left(\mathbf{R}^T\boldsymbol{\Sigma}^{-1}\mathbf{R}\right)^{-1}$ with $\mathbf{R} = \begin{bmatrix} \boldsymbol{\gamma} & \mathbf{j} & \mathbf{A} \end{bmatrix}$ and $\mathbf{j} = [1, 1 \ldots 1]^T$. Moreover the efficient frontier $\sigma(\mu)$, allowing for different constraints binding for different regions, is in general piecewise quadratic and

- *is defined over a closed interval in μ, say $I = [\mu_{\min}, \mu_{\max}]$*
- *is convex in μ; and*
- *has, except in pathological cases, continuous first (but not necessarily second) derivatives over its domain.*

This allows us to compute the efficient frontier in practice, given the expected return vector $\boldsymbol{\gamma}$, covariance matrix $\boldsymbol{\Sigma}$ and the constraints.