



Optimal investment strategies for the HARA utility under the constant elasticity of variance model

Eun Ju Jung, Jai Heui Kim*

Department of Mathematics, Pusan National University, Pusan 609-735, Republic of Korea

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ABSTRACT

We give an explicit expression for the optimal investment strategy, under the constant elasticity of variance (CEV) model, which maximizes the expected HARA utility of the final value of the surplus at the maturity time. To do this, the corresponding HJB equation will be transformed into a linear partial differential equation by applying a Legendre transform. And we prove that the optimal investment strategy corresponding to the HARA utility function converges a.s. to the one corresponding to the exponential utility function.

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1. Introduction

The managers of pension funds buy reinsurance and/or invest their company's surplus in a financial market to reduce the risk. Two of fundamental aims that an insurance company pursues is to minimize the ruin probability of the company and to maximize the expected utility of the final surplus at the maturity time T .

In the case of no reinsurance and no investment during the period $[0, T]$ before retirement, the surplus process $(V(t))_{t \in [0, T]}$ of the company will be described by the following form:

$$\begin{cases} dV(t) = \mu_0 dt \\ V(0) = V_0, \end{cases} \quad (1.1)$$

where the constant $V_0 > 0$ is the initial surplus and the constant $\mu_0 > 0$ is the continuous rate of contribution.

We assume that all of the surplus is invested in a financial market which consists of two securities, named B and S , whose prices are given by the following differential equations:

$$dB(t) = rB(t)dt \quad (1.2)$$

and

$$dS(t) = \mu S(t)dt + kS^{1+\gamma}(t)dW(t), \quad (1.3)$$

where r, μ, k and γ are some constants with $0 < r < \mu$ and $\gamma \leq 0$, and $(W(t))_{t \in [0, T]}$ is a standard Brownian motion on a complete probability space (Ω, \mathcal{F}, P) with a filtration $(\mathcal{F}_t)_{t \in [0, T]}$. Here r is a rate of return of the risk-free asset B , μ is an expected instantaneous rate of return of the risky asset S and γ is the elasticity parameter. In this case we call (B, S) a financial market with the constant elasticity variance (CEV) model.

We denote by $\beta(t)$ the proportion invested in the risky security S at time $t \in [0, T]$. We disallow leverage and short-sales. In this case it holds that $0 \leq \beta(t) \leq 1$ for all $t \in [0, T]$. Therefore, at any time $0 \leq t < T$, a nominal amount $V(t)(1 - \beta(t))$ is allocated to the risk-free asset B . We treat the proportion $\beta(t)$ of the surplus at time t as a control parameter. Then the surplus process $(V(t))_{t \in [0, T]}$ is given by the following stochastic differential equations:

$$\begin{cases} dV(t) = [V(t)\{\beta(t)(\mu - r) + r\} \\ + \mu_0]dt + V(t)k\beta(t)S^\gamma(t)dW(t) \\ V(0) = V_0. \end{cases} \quad (1.4)$$

Given a strategy $\beta(\cdot)$, the solution $(V^\beta(t))_{t \in [0, T]}$ of (1.4) is called the surplus process corresponding to $\beta(\cdot)$.

In this paper, we are interested in maximizing the expected hyperbolic absolute risk aversion (HARA) utility of the company's terminal surplus. The HARA utility function with parameters η, p and q is given by

$$U(v) = U(\eta, p, q; v) = \frac{1-p}{qp} \left(\frac{qv}{1-p} + \eta \right)^p, \quad (1.5)$$

$$q > 0, p < 1, p \neq 0.$$

* Corresponding author. Tel.: +82 51 510 2209; fax: +82 51 581 1458.
E-mail address: jaihkim@pusan.ac.kr (J.H. Kim).

We can check that

$$U(0, p, 1 - p; v) = U_{power}(v) \equiv \frac{1}{p}v^p$$

(power utility function) (1.6)

and

$$\lim_{p \rightarrow -\infty} U(1, p, q; v) = U_{exp}(v) \equiv -\frac{1}{q}e^{-qv}$$

(exponential utility function). (1.7)

In the case that $\gamma = 0$, Devolder et al. (2003) found an explicit expression for the optimal asset allocation which maximizes the expected power or exponential utility of the final annuity fund at retirement and at the end of the period after retirement. In the case that γ satisfies the more general condition $\gamma \leq 0$, by using the Legendre transform and dual theory, Xiao et al. (2007) solved the same problem as Devolder et al. (2003) with the logarithmic utility function defined as a limit of a family of modified HARA utility functions when $p \rightarrow 0$ (cf. Grasselli (2003)). And Gao (2009) extended the work by Devolder et al. (2003) to the case that $\gamma \leq 0$ by the same method as Xiao et al. (2007).

In this paper we find an explicit expression for the optimal investment strategy $\beta^*(\cdot)$, under the same condition that $\gamma \leq 0$ as in the works by Xiao et al. (2007) and Gao (2009), which maximizes the expected HARA or power utility of the final value of the surplus process given by the stochastic differential equation (1.4). And we prove that the optimal solution corresponding to the HARA utility function converges a.s. to the solution corresponding to the exponential utility function as $p \rightarrow -\infty$. Grasselli (2003) investigated these problems in a financial market model which consists with a risk-free asset whose short rate of return follows the CIR model and d risky assets without elasticities.

During the period $[T, T + N]$ after retirement, the surplus process is given by Eq. (1.4) such that μ_0 is replaced by the minus of the continuous rate of annuity benefit ($-\lambda_0$). This shows that our problems before and after retirement are mathematically the same. So we consider only the problem before retirement.

Xiao et al. (2007) and Gao (2009) derived the optimal strategy by solving the corresponding Hamilton–Jacobi–Bellman (HJB) equation. Usually, in stochastic optimal problem, the corresponding HJB equation is a nonlinear partial differential equation and it is difficult to solve. So they transformed the HJB equation into a linear partial differential equation by applying a Legendre transform. But Gu et al. (2010) gave an optimal reinsurance and investment strategy by directly solving the HJB equation with very complicated calculus. In this paper we use the same methods as the works by Xiao et al. (2007) and Gao (2009).

The structure of the paper is as follows. In Section 2 we formulate our problem and give a theory background based on the stochastic optimal control theory (Björk, 1998; Øksendal, 1998) and properties for a Legendre transform (Jonsson and Sircar, 2002; Xiao et al., 2007; Gao, 2009). In Section 3 we give an explicit expression for the optimal investment strategy corresponding to the HARA utility function. In Section 4, we give an explicit expression for the optimal investment strategy corresponding to the power utility function as a special case of the HARA class, and prove that the optimal investment strategy corresponding to the HARA utility function converges a.s. to the one corresponding to the exponential utility function.

2. Formulation of the problem and theory background

A control function $\beta(\cdot)$ in (1.4) is said to be admissible if $(\beta(t))_{t \geq 0}$ is a \mathcal{F}_t -adapted process satisfying $0 \leq \beta(t) \leq 1$ for all $t \in [0, T]$. The set of all admissible controls is denoted by \mathcal{A} .

We use the HARA utility function $U(v) = U(\eta, p, q; v)$ with parameters η, p and q defined by (1.5). For the surplus process $(V^\beta(t))_{t \in [0, T]}$ given by (1.4), put

$$J^\beta(t, s, v) = E[U(V^\beta(T)) | S(t) = s, V^\beta(t) = v] \tag{2.1}$$

for all $(t, s, v) \in [0, T] \times R^1 \times R^1$, where $E[X|A]$ is the conditional expectation of a random variable X given an event A . In stochastic optimal control theory it is important to find the optimal value function

$$H(t, s, v) = \sup_{\beta \in \mathcal{A}} J^\beta(t, s, v) \tag{2.2}$$

and the optimal strategy $\beta^*(\cdot)$ such that

$$J^{\beta^*}(t, s, v) = H(t, s, v). \tag{2.3}$$

In this paper we will give an explicit expression of $\beta^*(t)$. The following two theorems are essential to solve our problem. The proofs are standard and can be found in many text books (e.g. Björk (1998) and Øksendal (1998)).

Theorem 2.1 (HJB Equation). (1) Assume that $H(t, s, v)$ defined by (2.2) is twice continuously differentiable on $(0, \infty)$, i.e., $\in C^{1,2}$. Then $H(t, s, v)$ satisfies the following HJB equation:

$$\begin{cases} \sup_{\beta \in \mathcal{A}} L^\beta H(t, s, v) = 0 \\ H(T, s, v) = U(v) \end{cases} \tag{2.4}$$

for all $(t, s, v) \in [0, T] \times R^1 \times R^1$, where L^β is the infinitesimal generator corresponding to the diffusion process defined by the stochastic differential equation (1.4), i.e.,

$$\begin{aligned} L^\beta = & \frac{\partial}{\partial t} + \mu s \frac{\partial}{\partial s} + \{v[\beta(t)(\mu - r) + r] + \mu_0\} \frac{\partial}{\partial v} \\ & + \frac{1}{2} v^2 \beta(t)^2 k^2 s^{2\gamma} \frac{\partial^2}{\partial v^2} + \frac{1}{2} k^2 s^{2+2\gamma} \frac{\partial^2}{\partial s^2} \\ & + \beta(t) k^2 s^{1+2\gamma} v \frac{\partial^2}{\partial s \partial v}. \end{aligned} \tag{2.5}$$

(2) Let $G(t, s, v)$ be a solution of the HJB equation (2.4). Then the value function $H(t, s, v)$ to the control problem (2.2) is given by

$$H(t, s, v) = G(t, s, v).$$

Moreover if, for some control $\bar{\beta}(\cdot)$, $L^{\bar{\beta}}G(t, s, v) = 0$ for all $(t, s, v) \in [0, T] \times R^1 \times R^1$, then it holds $G(t, s, v) = J^{\bar{\beta}}(t, s, v)$. In this case $\bar{\beta}(t) = \beta^*(t)$ and $J^{\bar{\beta}}(t, s, v) = J^{\beta^*}(t, s, v)$.

By Theorem 2.1, the HJB equation associated with our optimization problem (2.2) and (2.3) is

$$\begin{aligned} 0 = & H_t + \mu s H_s + (rv + \mu_0)H_v + \frac{1}{2} k s^{2\gamma+2} H_{ss} \\ & + \sup_{\beta} \left\{ \beta(\mu - r)v H_v + \beta k^2 s^{2\gamma+1} v H_{sv} \right. \\ & \left. + \frac{1}{2} \beta^2 k^2 s^{2\gamma} v^2 H_{vv} \right\}, \end{aligned} \tag{2.6}$$

where $H_t, H_v, H_s, H_{vv}, H_{ss}, H_{sv}$ denote partial derivative of first and second orders with respect to time, stock price and wealth parameters. It is easy to show that the optimal strategy β^* is given by

$$\beta^* = -\frac{(\mu - r)H_v + k^2 s^{2\gamma+1} H_{sv}}{v k^2 s^{2\gamma} H_{vv}}. \tag{2.7}$$

Inserting (2.7) into (2.6), we obtain the following second order partial differential equation for the optimal value function H ;

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