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The optimal mean–variance investment strategy under value-at-risk constraints

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ABSTRACT

This paper is devoted to study the effects arising from imposing a value-at-risk (VaR) constraint in the mean–variance portfolio selection problem for an insurer who receives a stochastic cash flow which he must then invest in a continuous-time financial market. For simplicity, we assume that there is only one investment opportunity available for the insurer, a risky stock. Using techniques of stochastic linear–quadratic (LQ) control, the optimal mean–variance investment strategy with and without the VaR constraint is derived explicitly in closed forms, based on the solution of the corresponding Hamilton–Jacobi–Bellman (HJB) equation. Furthermore, a numerical example is proposed to show how the addition of the VaR constraint affects the optimal strategy.

1. Introduction

The mean–variance model of Markowitz (1952, 1959) is a cornerstone of modern portfolio theory. The most important contribution of this model is that it enables an investor to optimally select mean–variance efficient portfolios for seeking the highest return after specifying his acceptable risk level. Since Markowitz's pioneering work, the mean–variance model was extended from the single-period case to the multi-period discrete-time case (see Hakansson, 1971; Pliska, 1997; Samuelson, 1969, etc.) and the continuous time case during the past decades (see Cox and Huang, 1989; Duffie and Richardson, 1991; Karatzas et al., 1987; Schweizer, 1996, etc.). However, when studying these two kinds of dynamic portfolio selection models, most research works have been dominated by those of maximizing expected utility functions of the terminal wealth. Nevertheless, when using this approach, the tradeoff information between the risk and the expected return is implicit, which makes the investment decision less intuitive. In 2000, Zhou and Li introduce the stochastic linear–quadratic (LQ) control as a general framework to study the mean–variance optimization problem. Within this framework they have established a natural connection of the portfolio selection problems and standard stochastic control models and attained some elegant results for a continuous-time mean–variance model with determined coefficients.

When using the stochastic LQ control approach to deal with the continuous time mean–variance problem, the terminal wealth is a random variable with a distribution that is often extremely skewed and shows considerable probability in regions of small values of the terminal wealth. This means that the optimal terminal wealth may exhibit large shortfall risks. In order to prevent investors from extremely dangerous positions in the market, it is thus more reasonable to consider asymmetric risk measures, e.g. value-at-risk (VaR), to limit the exposure to market risks.

In market risk management, it is widely accepted that VaR is a useful summary measure of market risks which regulatory authorities sometimes enforced investors to use. VaR is actually the maximum expected loss over a given horizon period at a given level of confidence. However, VaR may have undesirable properties such as lack of sub-additivity; an alternative coherent risk measure, conditional value-at-risk (CVaR, or sometimes called expected shortfall), has been introduced to overcome these conceptual deficiencies of VaR. Since the insurance loss is often characterized as having a long right tail risk, to capture this feature, the tail conditional expectation (TCE) is proposed in the literature and used in practice, which is defined as the conditional expectation of

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losses above the VaR to measure the average along the right-tail. It should be noted that TCE coincides with CVaR for fixed level when the loss has a continuous distribution. In this case, TCE and CVaR are actually the weighted average of VaR, in other words, all of them are equivalent in this paper. For comprehensive introduction to risk management using VaR, CVaR or TCE, we refer the reader to Jorion (1997) or Acerbi and Tasche (2002).

Recognizing that risk management is typically not an investor's primary objective, the investors would like to limit their risks while maximizing expected utility. This leads to stochastic control problems under restrictions on such risk measures. There has been considerable interest in the study of portfolio selection models subject to a VaR constraint. Emmer et al. (2000), Alexander and Baptista (2004, 2007) investigate the optimal portfolio choices subject to a VaR or CVaR constraint in a static (one-period) setting. The similar problems in a dynamic setting have started to draw more attention recently; Basak and Shapiro (2001) focus on the optimal portfolio policies of a utility-maximizing agent by imposing the VaR constraint at one point in time. Cuoco et al. (2008) developed a realistic dynamically consistent model of the optimal behavior of a trader subject to risk constraints. They assume that the risk of the trading portfolio is re-evaluated dynamically by using the conditioning information, and hence the trader must satisfy the risk limit continuously. Yiu (2004) explicitly derived the standard VaR constraint on total wealth and obtained the optimal trading strategy (without consideration of re-insurance). Pirvu (2005) started with the model of Cuoco et al. (2008) and found the optimal growth portfolio subject to these risk measures. Pirvu (2007) extended those results by extensively studying the optimal investment and consumption strategies for both logarithmic utility and non-logarithmic CRRA utilities. Recently, Chen et al. (2010) have investigated the optimal investment–reinsurance policy for an insurance company subject to a VaR constraint when minimizing the ruin probability.

Motivated by Zhou and Li (2000) and Yiu (2004), this paper addresses the problem of an insurer who receives an uncontrollable stochastic cash flow which he must then invest in a continuous-time financial market in order to maximize the weighted average of the expectation and the variance of his terminal wealth at a horizon time. The main focus in this paper is on the mean–variance optimization problem of the insurer subject to a risk limit specified in terms of VaR on his future net worth. To our knowledge, this problem has not yet received a complete treatment in the existing literature. In this paper, we derive the optimal mean–variance investment strategy under a standard VaR constraint by solving the corresponding Hamilton–Jacobi–Bellman (HJB) equation and explore how the addition of a risk constraint affects the optimal solution.

The rest of the paper is organized as follows. Section 2 describes the model, including the definition of the VaR on the future net worth process. Section 3 contains the main characterization result of the VaR constraint and formulates the portfolio optimization problem that can be eventually discussed as a stochastic LQ problem. Section 4 gives the explicit solution of the optimal mean–variance strategy without the VaR constraint by solving the corresponding HJB equation. Section 5 discusses the optimal mean–variance strategy with the VaR constraint. Section 6 provides a numerical example to show how the addition of the VaR constraint affects the optimal strategy. Section 7 concludes the paper.

2. The model

2.1. The continuous-time investment model

All stochastic processes introduced below are supposed to be adapted in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $\mathcal{F}_t, t \geq 0$ is a filtration satisfying the usual conditions. Moreover, it is assumed throughout this paper that all inequalities as well as equalities hold P-almost surely.

Following the framework of Browne (1995), for simplicity, and without any loss of generality, we assume that there is only one risky stock available for investment, whose price at time t will be denoted by P_t which satisfies the following stochastic differential equation

$$dP_t = P_t(\mu dt + \sigma dW_t^{(1)}), \tag{1}$$

where $\mu > 0$ is the appreciation rate and $\sigma > 0$ is the volatility or the dispersion of the stock. $W_t^{(1)}$ is a standard Brownian motion.

Since we are concerned with investment behavior in the presence of a stochastic cash flow, or a liquid reserve denoted by Y_t, which is another Brownian motion with drift α and diffusion parameter $\beta > 0$,

$$dY_t = \alpha dt + \beta dW_t^{(2)}, \tag{2}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are possibly correlated with correlation coefficient ρ. In case there would only be one source of randomness left in the model, we also assume that $\rho^2 < 1$.

To motivate the model (2), it is convenient to start from the classical Cramer–Lundberg model in which the reserve (surplus) of the insurer is given by $\tilde{Y}_t = \tilde{Y}_0 + ct - \sum_{i=1}^{N_t} \xi_i$, where c is the amount of premium per unit time received by the insurer, $\{N_t\}$ is a Poisson process with intensity λ representing the arrival times of the claims and ξ_i (the claim sizes) are i.i.d. with common distribution having finite mean m and variance s^2. If we make change of time and normalize the state space: $\tilde{Y}_t \mapsto \frac{\tilde{Y}_{nt}}{\sqrt{n}}$, then the limiting process Y_t satisfies Eq. (2) with $\alpha = c - \lambda m$ and $\beta^2 = \lambda(m^2 + s^2)$. So Y_t is to be understood as the net claim process in the diffusion approximation model and the parameter α represents the relative safety loading of the claim process.

Now we will define an investment strategy f as an admissible adapted control process f_t, satisfying that $\int_0^T f_t^2 dt < \infty$, a.s., for all $T < \infty$. Note that f_t represents the amount invested in risky stock at time t, and we will not put more constraints on f_t. In particular, we will allow $f_t < 0$, which means that short-selling would be allowed; the insurer is also permitted to borrow money to buy stocks.

Assume that the trading takes place continuously and transaction cost is not considered. Therefore, following the investment strategy f, the wealth process of the insurer at time t, which will be denoted by X_t^f, can be given by the following stochastic differential equation with initial condition $X_0 = x$

$$dX_t = (f_t\mu + \alpha)dt + f_t\sigma dW_t^{(1)} + \beta dW_t^{(2)}, \qquad X_0 = x. \tag{3}$$

2.2. Value-at-risk

Now we want to introduce the definition of value-at-risk. Here we start by rewriting (3) into integration form

$$X_t^f = x + \int_0^t (f_s\mu + \alpha)ds + \int_0^t (f_s\sigma)dW_s^{(1)} + \int_0^t \beta dW_s^{(2)}, \tag{4}$$

where $x > 0$ denotes the initial value of the portfolio. Notice that (4) leads to

$$X_{t+\tau}^f = X_t + \int_t^{t+\tau} (f_s\mu + \alpha)ds + \int_t^{t+\tau} (f_s\sigma)dW_s^{(1)} + \int_t^{t+\tau} \beta dW_s^{(2)}, \tag{5}$$

for any $\tau > 0$.

If we assume that the investment strategy were kept constant during the time period $(t, t + \tau]$, i.e. $f_s \equiv f$, for any $s \in (t, t + \tau]$,