Applied Mathematics Letters 26 (2013) 908–912

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

# Optimal control and long-run dynamics for a spatial economic growth model with physical capital accumulation and pollution diffusion

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ARTICLE INFO

*Article history:*
Received 19 February 2013
Received in revised form 8 April 2013
Accepted 9 April 2013

*Keywords:*
Reaction–diffusion systems
Integral nonlocal term
Large-time behavior
Control problems
Economic growth
Environmental quality
Non-concave production function

ABSTRACT

In this work we analyze the large-time behavior of a spatially structured economic growth model coupling physical capital accumulation and pollution diffusion. This model extends other results in the literature along different directions. Alongside the classical Cobb–Douglas production function, a convex–concave production function is considered. We add a negative feedback to the production function in order to describe the (negative) influence of pollution on output, and therefore on capital accumulation. We also present an optimal control problem for the above model.



## 1. Introduction

Standard macroeconomics and environmental economics have been two completely independent research areas for a long time. Some recent work has started to develop a global theory combining these two branches of literature (see Brock and Taylor [1]). In this work, we continue this effort by introducing an economic geographical characterization and spatial variables into an optimal economic growth model, with the goal of analyzing the impact of both capital and pollution diffusion on the economic and environmental joint dynamics. Therefore, this work directly combines optimal growth theory with two other different strands present in the economic literature: economic geography (a topic of recent interest in the economic analysis literature) and environmental economics. The first studies in economic geography go back to Beckman [2] and Puu [3], who study regional problems based simply on flow equations. These works led to the development of a notion of economic geography that uses general equilibrium models to analyze the peculiarities of local and global markets, as well as the mobility of production factors (Krugman [4]; Fujita et al. [5]). More recently, this geographical approach has been introduced in economic growth models to study the connections between accumulation and diffusion of capital on economic dynamics (Brito [6]; Camacho and Zou [7]; Boucekkine et al. [8]; Capasso et al. [9]). The Solow model [10] with a

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http://dx.doi.org/10.1016/j.aml.2013.04.002

continuous spatial dimension has been extensively studied. Camacho and Zou [7] analyze problems of convergence across regions for when capital is mobile, while Brito [6] considers the case in which both capital and labor are mobile. Capasso et al. [9] introduce technology diffusion in the same model, under the scenario of a convex–concave production function. The Ramsey model [11] has been extended to a spatial dimension by Brito [6] and Boucekkine et al. [8], respectively in average and total utilitarianism versions. Other contributions which explore the spatial dimension in environmental and resource economics can be found in Brock and Xepapadeas [12], Xepapadeas [13], and Athanassoglou and Xepapadeas [14].

## 2. The model

Let $k(x,t)$ and $p(x,t)$ respectively denote the capital stock held by and the pollution stock faced by a representative household located at $x$ at date $t$, in a habitat $\overline{\Omega}$ (where $\Omega \subset R^N$ is taken as a nonempty and bounded domain with a smooth boundary), and $t \geq 0$. We also assume that the initial capital and pollution distribution, $k(x,0) = k_0(x)$ and $p(x,0) = p_0(x)$, are known and satisfy

$$k_0,\ p_0 \in L^\infty(\Omega), \quad k_0(x) \geq k_{00} > 0,\ p_0(x) \geq 0 \text{ a.e. } x \in \Omega \tag{H}$$

and there is no capital or pollution flow through the boundary of $\Omega$, namely that the normal derivatives $k_\nu(x,t) = p_\nu(x,t) = 0$ at $x \in \partial\Omega$ and $t \geq 0$. We assume a continuous space structure of both physical capital and pollution, so the model that we are interested in is the following:

$$\begin{cases} k_t(x,t) = d_1 \Delta k(x,t) + \dfrac{sf(k(x,t))}{1+p(x,t)^2} - \delta_1 k(x,t) - c(x,t)k(x,t), & (x,t) \in Q_{0,\infty} \\ p_t(x,t) = d_2 \Delta p(x,t) + \theta \displaystyle\int_\Omega f(k(x',t))\varphi(x',x)dx' - \delta_2 p(x,t), & (x,t) \in Q_{0,\infty} \end{cases} \tag{1}$$

subject to homogeneous Neumann boundary conditions

$$k_\nu(x,t) = p_\nu(x,t) = 0, \quad (x,t) \in \Sigma_{0,\infty}, \tag{2}$$

and initial conditions

$$k(x,0) = k_0(x), \qquad p(x,0) = p_0(x), \quad x \in \Omega. \tag{3}$$

The control variable $c(x,t)$ describes the level of consumption at the location $x$, at the time $t$ ($c \in L^\infty(\Omega \times (0,+\infty))$, $0 \leq c(x,t) \leq L$ a.e.), and $d_1, d_2, s, \theta, \delta_1, \delta_2, L$ are positive parameters. Here $Q_{a,b} = \Omega \times (a,b)$ and $\Sigma_{a,b} = \partial\Omega \times (a,b)$.

In the above model (1) the symbol $f$ denotes a production function; we assume that it is of the following form:

$$f(r) = \frac{\alpha_1 r^\gamma}{1+\alpha_2 r^\gamma}, \tag{4}$$

where $\alpha_1 \in (0,+\infty)$, $\alpha_2 \in [0,+\infty)$, $\gamma \in (0,+\infty)$.

For basic results concerning the solutions to reaction–diffusion systems without integral terms we refer the reader to [15].

We remark first that we deal here with a reaction–diffusion system with an integral (nonlocal) term. Let us notice that for $\alpha_2 = 0$ and $\gamma \in (0,1]$, we get the well known Cobb–Douglas production function. On the other hand, for $\alpha_2 > 0$ and $\gamma > 1$, we get an S-shaped production function. Skiba [16] is the first contribution in the economic literature dealing with non-concave or convex/concave production functions. From an economic perspective this kind of assumption is justified by empirical evidence from less developed countries. Finally, $\varphi$ is a kernel which satisfies the following hypotheses: $\varphi \in L^\infty(\Omega \times \Omega)$, and $\varphi(x',x) \geq 0$ a.e. $(x',x) \in \Omega \times \Omega$. For $\gamma \in [1,+\infty)$, via Banach's fixed point theorem and using the fact that $f$ is continuously differentiable, it is possible to prove that there exists a unique and nonnegative solution to (1)–(3) on the whole positive time semi-axis. Whenever $\gamma \in (0,1)$, $f$ is no longer differentiable at 0; however, since by $(H)$ we have that $k_0(x) \geq k_{00} > 0$ a.e. $x \in \Omega$, comparison results for parabolic equations and the fixed point theorem imply, in this case too, the existence and uniqueness of a nonnegative solution to (1)–(3), on the whole positive time semi-axis.

In Section 3 we analyze the qualitative behavior of system (1)–(3) for large times and for some relevant cases; due to page restrictions, the analysis of additional cases is left to a subsequent paper [17]. In the last section of the work we briefly discuss the following optimal control problem; we assume that there is a representative agent who wishes to maximize his or her inter-temporal utility subject to the dynamic constraints (1)–(3). The control problem reads as

$$\max_c \int_0^\infty \int_\Omega e^{-\rho t} \frac{[c(x,t)k(x,t)]^{1-\beta} - 1}{1-\beta} dx\, dt, \tag{5}$$

subject to (2)–(3); $\rho$ is a positive discount factor, $\beta \in [0,1)$ is a positive parameter and $\frac{[c(x,t)k(x,t)]^{1-\beta}-1}{1-\beta}$ is the CIES utility function. For the sake of simplicity we provide necessary optimality conditions for an approximating optimal control problem corresponding to $\beta = 0$.