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Automatica 44 (2008) 1471–1479
automatica
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Existence of perfect equilibria in a class of multigenerational stochastic games of capital accumulation☆

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Received 4 February 2008; accepted 13 March 2008

Abstract

In this paper we introduce a model of multigenerational stochastic games of capital accumulation where each generation consists of m different players. The main objective is to prove the existence of a perfect stationary equilibrium in an infinite horizon game. A suitable change in the terminology used in this paper provides (in the case of perfect altruism between generations) a new Nash equilibrium theorem for standard stochastic games with uncountable state space.

Keywords: Multigenerational stochastic game; Altruistic economy of growth; Perfect equilibrium; Capital accumulation; Hyperbolic discounting

1. Introduction

There is growing conviction that a framework of intergenerational altruism is needed to analyze various economic issues. Saez-Marti and Weibull (2005) noticed that "recognition of the economic importance of altruism goes back at least to Edgeworth (1881)". Ramsey (1928) used the current idea called "perfect altruism" in the realm of economic growth theory. Rawls (1971) postulated intergenerational altruism in order to develop his theory of justice in an intertemporal context. Implications of Rawlsian theory for a "just savings principle" were studied by Arrow (1973) and Dasgupta (1974). For a discussion of contexts in which altruism models have been explored the reader is also referred to Arrow (1999). Among other things, he mentions "the very original paper" by Phelps and Pollak (1968) where to analyze an altruistic growth (intergenerational strategic bequest problem) a deterministic intergenerational game was introduced. From a game-theoretic point of view, the model of Phelps and Pollak (1968) is an infinite horizon dynamic game with countably many identical short-lived players. Each player represents a generation which lives for one period and derives utility from its own consumption and that of its immediate successor. The solution concept associated with this kind of dynamic game is the Markov-stationary subgame-perfect equilibrium in pure strategies.

The existence of such an equilibrium in the case of deterministic transitions was established independently by Bernheim and Ray (1987) and Leininger (1986). Amir (1996a) first studied a model of strategic bequest with a stochastic transition function. His existence result is based on Schauder's fixed point theorem applied to some subclass of Lipschitz continuous stationary strategies. A different class of intergenerational games with stochastic transition function is studied in Nowak (2006a). The transition probability function is a convex combination of a finite family of probability measures depending on the state variable. Such a transition structure contains most examples studied in the literature, also in the context of standard nonzero-sum stochastic games, see Amir (1996b), Nowak (2003a) and Balbus and Nowak (2004). To obtain an equilibrium (that is a fixed point of some continuous mapping) Nowak (2006a) enlarges the set of stationary strategies by considering randomized ones, endowed with some natural compact topology. It turns out, however, that the fixed point is a pure stationary equilibrium.

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Editor Alain Haurie.

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0005-1098/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2008.03.001

The above-mentioned papers (and their references) concern so-called intergenerational games with paternalistic altruism between successive generations where the utility of each generation depends on its own extraction and that of its immediate successor only. Recently Haurie (2005) and Haurie (2006) studied interesting models of piecewise deterministic multigenerational dynamic games involving nonpaternalistic altruism. Adopting the idea of discounting introduced by Phelps and Pollak (1968) he presented interesting applications of subgame-perfect (stationary) equilibria in multigenerational dynamic games to analyze sustainable development or global climate change economic impact assessment. It was already known to Phelps and Pollak (1968) that a stationary equilibrium in an altruistic growth model may be Pareto-inefficient. However, as emphasized by Haurie (2005) such an equilibrium is *time consistent* which is very important from the point of view of many possible applications.

An interesting generalization of the model due to Phelps and Pollak (1968) to a stochastic game framework was given by Alj and Haurie (1983). They assumed that every generation consists of m different players. Thus, there are m different *families* or *dynasties* spanning over the sequence of all generations. A member of a given generation is the representative of one dynasty (or family). The state and action spaces for the players in the model of Alj and Haurie (1983) are finite. In this paper, we define and study multigenerational stochastic games of capital accumulation. We adopt the idea of Alj and Haurie (1983) in order to extend the usual stochastic game model of capital accumulation studied by Amir (1996b), Balbus and Nowak (2004) and Nowak (2003a) and many others. The state and action spaces in the considered game model are uncountable which makes the analysis more complicated in comparison with that of Alj and Haurie (1983). We accept assumptions (mainly on the transition structure) formulated first in Nowak (2006a), but only for one player in each generation. The idea of the proof also relies on using randomized (in some sense) strategies for the players which belong to a compact convex space. Our main purpose is to prove the existence of a stationary equilibrium in an infinite horizon multigenerational stochastic game. It turns out, however, that a pure equilibrium is obtained (Theorem 1). A key and new component of the proof (compared with that of Nowak (2006a)) is the uniqueness of Nash equilibrium in every one-step nonzero-sum game naturally arising from our analysis of the problem (Proposition 2).

In Section 2, we describe our model and state a theorem on stationary equilibria in an infinite horizon game. Section 3 contains important facts on the uniqueness of Nash equilibria in some special classes of static games. Section 4 contains the remaining details of the proof of our main result. In Section 5, we study finite horizon multigenerational games. A class of games with different moods of play is studied in Section 6.

2. Perfect equilibria in an infinite horizon game

The multigenerational stochastic game considered in this paper is a generalization of the growth model of Phelps and Pollak (1968) and a stochastic game of capital accumulation studied in Amir (1996b) and Nowak (2003a) and their references. The first paper on multigenerational stochastic games is due to Alj and Haurie (1983) where finite state and action spaces are considered. In economic growth models and stochastic games of capital accumulation the set of states is uncountable which leads to more difficult issues.

Let $T = \{1, 2, \ldots\}$ be the set of stages of the game. With any $t \in T$ is associated a *generation* $G_t := \{1_t, 2_t, \ldots, m_t\}$ of m players. It is assumed that $i_{t+1} \in G_{t+1}$ is a *descendant* of $i_t \in G_t$. Therefore, one can say that $F_i := \{i_t\}_{t\in T}$ is a *dynasty* or a *family* of players from the successive generations. Clearly, i_t denotes the i-th player in generation G_t.

The *multigenerational stochastic game* is defined by the objects: $S, \{A_i(s)\}_{s\in S}, u_i, q, \alpha, \beta$ satisfying the following assumptions:

A1: S is an interval in $R_+ = [0, \infty)$ containing zero and is called the *state space*, or the set of all possible *capital stocks*.

A2: $A_i(s)$ is the *set of actions* or *extraction possibilities* available to every player i_t from dynasty F_i in state $s \in S$. It is assumed that $A_i(s) = [0, a_i(s)]$ where $a_i : S \to R_+$ is a continuous nondecreasing function such that $a_i(0) = 0$. Moreover, for every $s \in S$,

$$a_1(s) + a_2(s) + \cdots + a_m(s) \le s.$$

Let

$$A(s) := A_1(s) \times A_2(s) \times \cdots \times A_m(s),$$
$$C := \{(s, x) : s \in S, x \in A(s)\}.$$

A3: For every $i_t \in G_t$, $u_{i_t} : R_+ \mapsto R$ is a nonnegative, bounded, increasing, twice continuously differentiable strictly concave function such that $u_{i_t}(0) = 0$. It is assumed that $u_{i_t} = u_i$ is the same for each $i_t \in F_i$ and is called the *instantaneous utility function* for every player $i_t \in F_i$.

At any period $t \in T$ of the game and for each $s_t \in S$, the instantaneous utility of player $i_t \in G_t$ is $u_i(x_{i_t})$, that is, it depends only on his extraction $x_{i_t} \in A_i(s_t)$.

A4: q is a transition probability from C to S, called the *law of motion* among states. If s_t is a state on some stage t of the game and the players from generation G_t select an $x_t \in A(s_t)$, then $q(\cdot|s_t, x_t)$ is the probability distribution of the next state (capital stock) s_{t+1}. It is also assumed that $q(\{0\}|0, 0) = 1$, and for any $s \in S$, $x = (x_1, x_2, \ldots, x_m) \in A(s)$,

$$q(\cdot|s, x) := (1 - g(s - \overline{x}))\nu(\cdot) + g(s - \overline{x})\mu(\cdot),$$

where $\overline{x} := \sum_{j=1}^{m} x_j$, $g : S \mapsto [0, 1]$ is a twice continuously differentiable, strictly concave, and increasing function, and μ, ν are probability measures on S.

A5: $\alpha : S \mapsto (0, 1]$ is a continuous function. For every $t \in T$ and $s_t \in S$, $\alpha(s_t)$ is called the *factor of altruism* (in state s_t) of every player $i_t \in G_t$ towards his descendants.

A6: $\beta \in (0, 1)$ and is called the *discount rate*.

Remark 1. When $S = [0, 1]$, $A_i(s) = [0, s/m]$, typical examples of functions g are: