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Leader–follower stochastic differential game with asymmetric information and applications[☆]

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ABSTRACT

This paper is concerned with a leader–follower stochastic differential game with asymmetric information, where the information available to the follower is based on some sub-σ-algebra of that available to the leader. Such kind of game problem has wide applications in finance, economics and management engineering such as newsvendor problems, cooperative advertising and pricing problems. Stochastic maximum principles and verification theorems with partial information are obtained, to represent the Stackelberg equilibrium. As applications, a linear–quadratic leader–follower stochastic differential game with asymmetric information is studied. It is shown that the open-loop Stackelberg equilibrium admits a state feedback representation if some system of Riccati equations is solvable.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}^n$ the Euclidean space of n-dimensional vectors, by $\mathbb{R}^{n\times d}$ the space of $n \times d$ matrices, by $\mathcal{S}^n$ the space of $n \times n$ symmetric matrices. $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the scalar product and norm in the Euclidean space, respectively. $\top$ appearing in the superscripts denotes the transpose of a matrix. f_x, f_{xx} denote the partial derivative and twice partial derivative with respect to x for a differentiable function f.

1.1. Motivation

First, we present two examples which motivate us to study leader–follower stochastic differential games with asymmetric information in this paper.

Example 1.1 (*Continuous Time Newsvendor Problem*). Let $D(\cdot)$ be the demand rate for a product in the market, which satisfies the stochastic differential equation (SDE)

$$\begin{cases} dD(t) &= a(\mu - D(t))dt + \sigma dW(t) + \widetilde{\sigma}\widetilde{W}(t), \\ D(0) &= d_0 \in \mathbf{R}, \end{cases}$$

where $a, \mu, \sigma, \widetilde{\sigma}$ are constants. We consider the market consisting of a manufacturer selling the product to end users through a retailer. At time t, the retailer chooses an order rate $q(t)$ for the product and decides its retail price $R(t)$, and is offered a wholesale price $w(t)$ by the manufacturer. We assume that items can be salvaged at unit price $S \geq 0$, and that items cannot be stored, i.e., they must be sold instantly or salvaged.

The retailer will obtain an expected profit

$$J_1\big(q(\cdot), R(\cdot), w(\cdot)\big) = \mathbb{E}\int_0^T \big[(R(t) - S)\min[D(t), q(t)] - (w(t) - S)q(t)\big]dt.$$

When the manufacturer has a fixed production cost per unit $M \geq 0$, he will get an expected profit

$$J_2\big(q(\cdot), R(\cdot), w(\cdot)\big) = \mathbb{E}\int_0^T (w(t) - M)q(t)dt.$$

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Let $\mathcal{F}_t$ denote the σ-algebra generated by Brownian motions $W(s), \widetilde{W}(s), 0 \le s \le t$. Intuitively $\mathcal{F}_t$ contains all the information up to time t. We assume that the information $\mathcal{G}_{1,t}, \mathcal{G}_{2,t}$ available to the retailer and the manufacturer at time t, respectively, are both sub-σ-algebras of $\mathcal{F}_t$. Moreover, the information available to them at time t is asymmetric and $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t}$. This can be explained from the practical application's aspect. Specifically, the manufacturer chooses a wholesale price $w(t)$ at time t, which is a $\mathcal{G}_{2,t}$-adapted stochastic process. And the retailer chooses an order rate $q(t)$ and a retail price $R(t)$ at time t, which are $\mathcal{G}_{1,t}$-adapted stochastic processes. For any $w(\cdot)$, to select a $\mathcal{G}_{1,t}$-adapted process pair $(q^*(\cdot), R^*(\cdot))$ for the retailer such that

$$\begin{aligned} J_1(q^*(\cdot), R^*(\cdot), w(\cdot)) &\equiv J_1\big(q^*(\cdot; w(\cdot)), R^*(\cdot; w(\cdot)), w(\cdot)\big) \\ &= \max_{q(\cdot), R(\cdot)} J_1(q(\cdot), R(\cdot), w(\cdot)), \end{aligned}$$

and then to select a $\mathcal{G}_{2,t}$-adapted process $w^*(\cdot)$ for the manufacturer such that

$$\begin{aligned} J_2(q^*(\cdot), R^*(\cdot), w^*(\cdot)) &\equiv J_2\big(q^*(\cdot; w^*(\cdot)), R^*(\cdot; w^*(\cdot)), w^*(\cdot)\big) \\ &= \max_{w(\cdot)} J_2\big(q^*(\cdot; w(\cdot)), R^*(\cdot; w(\cdot)), w(\cdot)\big), \end{aligned}$$

formulates a leader–follower stochastic differential game with asymmetric information. In this setting, the retailer is the follower and the manufacturer is the leader. Any process triple $(q^*(\cdot), R^*(\cdot), w^*(\cdot))$ satisfying the above two equalities is called an open-loop Stackelberg equilibrium. In Øksendal, Sandal, and Ubøe (2013), a time-dependent newsvendor problem with time-delayed information is solved, based on stochastic differential game (with jump–diffusion) approach. But it cannot cover our model.

Example 1.2 (*Cooperative Advertising and Pricing Problem*)**.** In supply chain management of the market, there are usually two members, the manufacturer and the retailer. Cooperative advertising is an important instrument for aligning manufacturer and retailer decisions. Specifically, we introduce the following SDE, which is the generalization of Sethi's stochastic sales-advertising model introduced by (He, Prasad, & Sethi, 2009):

$$\begin{cases} dx(t) = \big[\rho u(t)\sqrt{1-x(t)} - \delta x(t)\big]dt \\ \qquad\qquad + \sigma(x(t))dW(t) + \widetilde{\sigma}(x(t))d\widetilde{W}(t), \\ x(0) = x_0 \in [0,1], \end{cases}$$

where $x(t)$ represents the awareness share, i.e., the number of aware (or informed) customers expressed as a fraction of the total market at time t, ρ is a response constant, and δ determines the rate at which potential consumers are lost. $\sigma(x), \widetilde{\sigma}(x)$ are functions satisfying usual conditions.

At time $t \ge 0$, the manufacturer decides on the wholesale price $w(t)$ and the cooperative participation rate $\theta(t)$, and the retailer decides the channel's total advertising effort level $u(t)$ and the retail price $p(t)$. The sequence of the events is as follows. At time t, first, the manufacturer announces his wholesale price $w(t)$ and participation rate $\theta(t)$. Second, the retailer sets his retail price $p(t)$ and advertising effort rate $u(t)$ as his optimal response to the manufacturer's announced decisions. The retailer accomplishes this by solving an optimization problem to maximize his expected profit

$$\begin{aligned} J_1(w(\cdot), \theta(\cdot), u(\cdot), p(\cdot)) = \mathbb{E}\int_0^T e^{-rt}\big[&(p(t)-w(t)) \\ &\times D(p(t))x(t) - (1-\theta(t))u^2(t)\big]dt, \end{aligned}$$

where $r > 0$ is the discount rate and $0 \le D(p) \le 1$ is the demand function satisfying usual conditions. The manufacturer anticipates the retailer's reaction functions and incorporates them into his optimal control problem, and solves for his wholesale price policy $w(t)$ and the participation rate policy $\theta(t)$ at time t. Therefore, the manufacturer's optimization problem is to maximize his expected profit

$$\begin{aligned} &J_2(w(\cdot), \theta(\cdot), u(\cdot), p(\cdot)) \\ &= \mathbb{E}\int_0^T e^{-rt}\big[(w(t)-c)D(p(t))x(t) - \theta(t)u^2(t)\big]dt, \end{aligned}$$

where $c \ge 0$ is the constant unit production cost.

Define $\mathcal{F}_t := \sigma\{W(s), 0 \le s \le t\} \bigvee \sigma\{\widetilde{W}(s), 0 \le s \le t\}$. In the game setting, we assume that the information $\mathcal{G}_{1,t}, \mathcal{G}_{2,t}$ available to the retailer and the manufacturer at time t, respectively, are both sub-σ-algebras of the complete information filtration $\mathcal{F}_t$. Moreover, the information available at time t to them is asymmetric and $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t}$. In detail, the wholesale price $w(\cdot)$ and the participation rate $\theta(\cdot)$ of the manufacturer are $\mathcal{G}_{2,t}$-adapted processes. For the retailer, his advertising effort level $u(\cdot)$ and retail price $p(\cdot)$ are to be $\mathcal{G}_{1,t}$-adapted processes. For any $(w(\cdot), \theta(\cdot))$, first, to select a suitable process pair $(u^*(\cdot), p^*(\cdot))$ for the retailer such that

$$\begin{aligned} &J_1(w(\cdot), \theta(\cdot), u^*(\cdot), p^*(\cdot)) \\ &\equiv J_1\big(w(\cdot), \theta(\cdot), u^*(\cdot; (w(\cdot), \theta(\cdot))), p^*(\cdot; (w(\cdot), \theta(\cdot)))\big) \\ &= \max_{u(\cdot), p(\cdot) \ge 0} J_1(w(\cdot), \theta(\cdot), u(\cdot), p(\cdot)), \end{aligned}$$

and then to select a suitable process pair $(w^*(\cdot), \theta^*(\cdot))$ for the manufacturer such that

$$\begin{aligned} &J_2(w^*(\cdot), \theta^*(\cdot), u^*(\cdot), p^*(\cdot)) \\ &\equiv J_2\big(w^*(\cdot), \theta^*(\cdot), u^*(\cdot; (w^*(\cdot), \theta^*(\cdot))), p^*(\cdot; (w^*(\cdot), \theta^*(\cdot)))\big) \\ &= \max_{w(\cdot), 0 \le \theta(\cdot) \le 1} J_2\big(w(\cdot), \theta(\cdot), u^*(\cdot; (w(\cdot), \theta(\cdot))), \\ &\qquad p^*(\cdot; (w(\cdot), \theta(\cdot)))\big), \end{aligned}$$

formulates a leader–follower stochastic differential game with asymmetric information. See also He et al. (2009) for more details about cooperative advertising and pricing models in dynamic stochastic supply chains, where feedback Stackelberg equilibrium is obtained applying dynamic programming approach for stochastic differential game. However, the asymmetric information was not considered there.

1.2. Problem formulation

Inspired by the examples above, we study leader–follower stochastic differential games with asymmetric information in this paper.

Let $0 < T < \infty$ be a finite time duration and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. $(W(\cdot), \widetilde{W}(\cdot))$ is a standard $\mathbb{R}^{d_1+d_2}$-valued Brownian motion. Let $\{\mathcal{F}_t\}_{0 \le t \le T}$ be the natural augmented filtration generated by $(W(\cdot), \widetilde{W}(\cdot))$ and $\mathcal{F}_T = \mathcal{F}$.

Let the state of the system be described by the SDE

$$\begin{cases} dx^{u_1,u_2}(t) = b\big(t, x^{u_1,u_2}(t), u_1(t), u_2(t)\big)dt \\ \qquad\qquad + \sigma\big(t, x^{u_1,u_2}(t), u_1(t), u_2(t)\big)dW(t) \\ \qquad\qquad + \widetilde{\sigma}\big(t, x^{u_1,u_2}(t), u_1(t), u_2(t)\big)d\widetilde{W}(t), \\ x^{u_1,u_2}(0) = x_0, \end{cases} \tag{1}$$

where $u_1(\cdot)$ and $u_2(\cdot)$ are control processes taken by the two players in the game, labeled 1 (the follower) and 2 (the leader), with values in nonempty convex sets $U_1 \subseteq \mathbb{R}^{m_1}$, $U_2 \subseteq \mathbb{R}^{m_2}$, respectively. $x^{u_1,u_2}(\cdot)$, the solution to SDE (1) with values in $\mathbb{R}^n$, is the corresponding state process with initial state $x_0 \in \mathbb{R}^n$. Here $b(t, x, u_1, u_2) : \Omega \times [0,T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^n$, $\sigma(t, x, u_1, u_2) : \Omega \times [0,T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^{n \times d_1}$, $\widetilde{\sigma}(t, x, u_1, u_2) : \Omega \times [0,T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^{n \times d_2}$ are given $\mathcal{F}_t$-adapted processes, for each (x, u_1, u_2).