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Arbitrary real-order cost functions for signals and systems

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ABSTRACT

This work focuses on a new type of cost function based on fractional operators. To do so, the concept of definite integral is extended to arbitrary real-order. Some properties of this new fractional-order definite integral are studied and a fractional-order Barrow's rule is proposed. It is illustrated by an example (the design of an IIR filter) how this new kind of cost function can be a valuable tool in problems where optimal design methods are involved.

1. Introduction

Fractional-order calculus is the extension of integration and differentiation of integer-order to arbitrary real- or even complex-order [1–3], allowing calculations such as deriving a function to $\frac{1}{2}$ order.

Although Leibniz and L'Hôpital discussed the possibility that the n-th derivate ($d^n y/dx^n$) could be a fraction and not necessary integer in 1695, this branch of mathematical analysis was really developed at the beginning of the 19th century by Liouville, Riemann, Letnikov and others [4]. Thus, this type of calculus has its origin in an extension of meaning. Another mathematical example of extension of meaning is the calculus of factorial of integers to factorial of complex numbers by Euler's gamma function $\Gamma(z)$, which is also one of the basic functions of fractional calculus [5,6].

Two of the most important definitions of the fractional derivatives were given by Riemann–Liouville—RL—(1), generally used in continuous domain, and Grünwald–Letnikov—GL—(2), used in discrete calculations. For a wide class of functions, which appear in real physical and engineering applications, these definitions are equivalent if $f(t)$ has $n+1$ continuous derivatives for $t \geq 0$. Therefore, RL is usually used for algebraic manipulations and GL for numerical integration and simulation [6].

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \quad n \leq \alpha \leq n+1 \tag{1} $$

$$ D^{\alpha} f(t)_{t=kh} = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(kh-jh) \tag{2} $$

On the other hand, RL fractional integration is given by

$$ I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t>0, \quad \alpha \in \mathbb{R}^+ \tag{3} $$

and GL fractional integration is just derivation for negative values of α.

One of the most important consequences of these definitions is that the discretization of fractional-order operators has infinite memory [7] in accordance with

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expression (2). However, for practical applications this fact is not in general a problem. Due to the coefficients in the GL definition, only the "recent past" plays an important role in evaluating D^{α} and the series in (2) can be truncated. This is the so-called short memory principle [6].

In the rest of this work we shall be interested in the discrete domain and therefore it will be assumed that only the GL definition is used (the RL definition has just been given in order to complete this introduction).

Although during the last 300 years the fractional derivation theory has been a theoretical field of mathematics, in recent years this branch of mathematical analysis has given rise to many applications in several fields. For example and without the intention of being exhaustive, we can mention the prediction of Great Salt Lake elevation in Utah [8], the modelling of fluid particle transverse accelerations [9], and a fractional model for the ultracapacitor [10] as modelling examples; for physical applications, an excellent review is given in [11]; a fractional calculus approach to viscoelasticity is given in [12]; for automatic control, fractional order controllers are used to enhance the system performance [13,14]; or for signal processing, we can find works both for finite impulse response (FIR) [16] and infinite impulse response (IIR) [17,18] filter design.

In those fields it is common to solve optimization problems for designing or identification purposes. The aim is to determine a set of parameters, such that the value of an objective function or optimization index is optimized subject to a set of possible constraints. In this context, in [15] the concept of optimization index is generalized to fractional-order for designing electronics circuits using genetic algorithms.

In this paper we propose the generalization of the cost function

$$J = \int_a^b [f(x)]^2\, dx \tag{4}$$

to arbitrary real-order:

$$J_{Frac} = \int_a^b D^{1-\alpha}[f(x)]^2\, dx, \quad \alpha,a,b \in \mathbb{R} \tag{5}$$

(The latter expression has been successfully used in the field of automatic control for designing predictive controllers [19].) In the following a study of the mathematical meaning of (5) will be carried out. To do so, the concept of definite integral will be extended to arbitrary real-order and a generalization of Barrow's rule will be proposed for fractional calculus.

It has been proved in literature that fractional-order filters outperform the integer ones (see, for instance, [20]). In this paper a different approach is proposed: It will be shown with an example how an IIR filter designed using a fractional-order cost function like (5) can outperform its integer counterpart as well.

This paper is organized as follows: In the next section some mathematical background related to fractional calculus is enunciated. In Section 3 the main result is given. The concept of fractional-order definite integral is introduced and its main properties are analysed. A numerical method to evaluate expressions like (5) is given as well. In Section 4 the example is developed. Finally, Section 5 draws the main conclusions of this work.

2. Background

In this section we enunciate some well-known results that will be needed in the rest of this work. Proofs can be found in [21].

Our first concern is with constant functions. There are functions that are effectively a constant with regard to a certain fractional derivative. Let $C(\alpha)$ be the generic constant of index α. For instance, $f(x)=x^{-1/2}$ is a $C(1/2)$ constant as $D^{1/2}(x^{-1/2})=0$.

Theorem 2.1. *Let $f(x)$ be a function having a power series representation and assume that there exist derivatives $D^{\mu}[f(x)]$, $D^{\nu}[f(x)]$, and $D^{\alpha}[f(x)]$ with $\alpha=\mu+\nu$. If $f(x)$ is not a $C(\mu)$ and $C(\nu)$ constant then we have*

$$D^{\alpha}[f(x)] = D^{\mu+\nu}[f(x)] = D^{\mu}[D^{\nu}[f(x)]] = D^{\nu}[D^{\mu}[f(x)]] \tag{6}$$

Secondly, it is important to note that in the real indexed fractional derivative formalism the integral is only a particular case, with a negative integer value

$$D^{-1}[f(x)] = \int_0^x f(\tau)\, d\tau \tag{7}$$

The generalization is straightforward, being $D^{-\alpha}$ the integral of arbitrary order [1,6]. In the following we will use the notation $I^{\alpha}\equiv D^{-\alpha}$, $\alpha \in \mathbb{R}$ (assuming that only the GL definition is considered).

Theorem 2.2. *Let $f(x)$ be a function $C(-1)$ constant. In this case we have $f(x)\equiv 0$.*

(*This theorem simply states that the only function that satisfies $D^{-1}[f(x)] = 0 = \int_0^x f(u)\, du$, $\forall x \in \mathbb{R}$,* is $f(x)\equiv 0$.)

3. Main result

3.1. Fractional-order definite integral

Theorem 3.1. *Let $f(x)$ be a function with a power series representation and assume that f is not a $C(1-\alpha)$ constant, with $\alpha \in \mathbb{R}$. Then*

$$\int_a^b [D^{1-\alpha} f(x)]\, dx = F^{\alpha}(b) - F^{\alpha}(a) \tag{8}$$

where F^{α} is any primitive function of order α of $f(x)$.

Proof. We first transform the operator D into I:

$$\int_a^b [D^{1-\alpha} f(x)]\, dx = \int_a^b [I^{\alpha-1} f(x)]\, dx \tag{9}$$

with $\alpha \in \mathbb{R}$ (for the GL definition). This expression can be expanded using Barrow's rule for the integer-order definite integral:

$$\int_a^b [I^{\alpha-1} f(x)]\, dx = \int [I^{\alpha-1} f(x)]\, dx|_{x=b} - \int [I^{\alpha-1} f(x)]\, dx|_{x=a} = I^1 I^{\alpha-1} f(x)|_{x=b} - I^1 I^{\alpha-1} f(x)|_{x=a} \tag{10}$$

If f is not a $C(1-\alpha)$ constant (from Theorem 2.2 it is not necessary to assure that f is not a $C(-1)$ constant