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Ruin with insurance and financial risks following the least risky FGM dependence structure

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ABSTRACT

Recently, Chen (2011) studied the finite-time ruin probability in a discrete-time risk model in which the insurance and financial risks form a sequence of independent and identically distributed random pairs with common bivariate Farlie–Gumbel–Morgenstern (FGM) distribution. The parameter θ of the FGM distribution governs the strength of dependence, with a smaller value of θ corresponding to a less risky situation. For the subexponential case with $-1 < \theta \leq 1$, a general asymptotic formula for the finite-time ruin probability was derived. However, the derivation there is not valid for the least risky case $\theta = -1$. In this paper, we complete the study by extending it to $\theta = -1$. The new formulas for $\theta = -1$ look very different from, but are intrinsically consistent with, the existing one for $-1 < \theta \leq 1$, and they offer a quantitative understanding on how significantly the asymptotic ruin probability decreases when θ switches from its normal range to its negative extremum.

1. Introduction

Consider a discrete-time insurance risk model. Within period i, $i \in \mathbb{N}$, the net insurance loss (equal to the total claim amount minus the total premium income) is denoted by a real-valued random variable X_i. Suppose that the insurer makes both risk-free and risky investments, leading to an overall stochastic discount factor, denoted by a nonnegative random variable Y_i, over the same time period. Here $\{X_i, i \in \mathbb{N}\}$ and $\{Y_i, i \in \mathbb{N}\}$ are often called insurance risks and financial risks, respectively, in the recent literature since Norberg (1999) and Tang and Tsitsiashvili (2003). The sum

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \quad n \in \mathbb{N}, \tag{1.1}$$

represents the stochastic present value of aggregate net losses up to time n. As usual, the probability of ruin by time n is defined to be

$$\psi(x; n) = \Pr\left(\max_{1 \leq m \leq n} \sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j > x \right), \quad n \in \mathbb{N}, \tag{1.2}$$

where $x \geq 0$ is the initial risk reserve of the insurer. Relation (1.2) serves as an efficient platform for the study of the interplay of insurance and financial risks causing the insurer's insolvency. See Nyrhinen (1999, 2001, 2012) and Tang and Tsitsiashvili (2003, 2004) for more background information.

Recently, Chen (2011) studied the asymptotic behavior of the ruin probability $\psi(x; n)$ in (1.2) for the case with dependent insurance and financial risks. Precisely, it was assumed that (X_i, Y_i), $i \in \mathbb{N}$, form a sequence of independent and identically distributed (i.i.d.) copies of a generic random pair (X, Y) whose components are however dependent. The dependence between X and Y was realized via a bivariate Farlie–Gumbel–Morgenstern (FGM) distribution of the form

$$\Pi(x, y) = F(x)G(y)\left(1 + \theta \overline{F}(x)\overline{G}(y)\right), \tag{1.3}$$

where $F = 1 - \overline{F}$ on $\mathbb{R} = (-\infty, \infty)$ and $G = 1 - \overline{G}$ on $\mathbb{R}_+ = [0, \infty)$ are marginal distributions of (X, Y), and $\theta \in [-1, 1]$ is a parameter governing the strength of dependence. Under the assumptions that F is a subexponential distribution, G fulfills some constraints in order for the product convolution of F and G (see (2.1)) to be a subexponential distribution too, and $\theta \in (-1, 1]$, Chen (2011) derived a general asymptotic formula for $\psi(x; n)$. Note that the assumption $\theta \neq -1$ was essentially applied there; see related discussions on

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p. 1041 of Chen (2011). Hence, the derivation of Chen (2011) is not valid for $\theta = -1$.

The FGM distribution (1.3) describes an asymptotically independent situation. Recall that, for a copula function $C(\cdot, \cdot)$ on $(0, 1)^2$, its survival copula is defined as $\hat{C}(u, v) = u + v - 1 + C(1-u, 1-v)$. For the FGM case, we have

$$\hat{C}(u, v) = C(u, v) = uv\left(1 + \theta(1-u)(1-v)\right), \quad (u, v) \in (0, 1)^2.$$

For every $\theta \in [-1, 1]$, the coefficient of upper tail dependence is

$$\chi = \lim_{u\downarrow 0} \frac{\hat{C}(u, u)}{u} = 0.$$

See Section 5.2 of McNeil et al. (2005) for details of the concepts used here. Nevertheless, asymptotically independent random variables may still exhibit different degrees of dependence. In this regard, Coles et al. (1999) proposed to use

$$\hat{\chi} = \lim_{u\downarrow 0} \frac{2\log u}{\log \hat{C}(u, u)} - 1$$

to measure more subtly the strength of dependence in the asymptotic independence case. With a bit of calculation, we see that $\hat{\chi} = 0$ for $\theta \in (-1, 1]$ while $\hat{\chi} = -1/3$ for $\theta = -1$. This illustrates the essential difference between the cases $-1 < \theta \le 1$ and $\theta = -1$.

In this paper we still look at the ruin probability (1.2) but for the case $\theta = -1$. It turns out that, not surprisingly though, the asymptotic behavior of $\psi(x; n)$ in the case $\theta = -1$ is very different from that in the case $-1 < \theta \le 1$. Due to the distinction between the two cases, in the present study new technicalities will be needed and more precise asymptotic analysis will be conducted. The main difficulty exists in dealing with the tail behavior of the product of X and Y following the FGM structure (1.3) with $\theta = -1$. Recent related discussions on the product of heavy-tailed (and dependent) random variables can be found in Hashorva et al. (2010), Jiang and Tang (2011), Yang et al. (2011), Yang and Hashorva (2013), and Yang and Wang (2013), among others.

While the scientific value of the present study is revealed during solving a series of technical problems to complement a previous study, we would like to stress its practical relevance in insurance and finance. When the insurance business goes insolvent, the insurer will of course become more conservative with investments. Moreover, stock market crashes will certainly increase the prudence of not only banking but also insurance regulators. These require the insurer to observantly adjust between the insurance and financial markets, leading to negatively dependent insurance and financial risks. Note that under the FGM framework $\theta = -1$ exhibits an extremely negative, thus the least risky, scenario in which hypothetically the insurer complies with the most conservative self-adjustment mechanism. The ruin probability for $\theta = -1$ should be smaller, at least asymptotically, than that for $-1 < \theta \le 1$, as is confirmed by our main results in Section 3. Thus, the present study devoted to the least risky scenario of the FGM framework offers some new insight into the insolvency of the insurance business in the presence of dependent insurance and financial risks.

The rest of this paper consists of four sections. Section 2 prepares preliminaries of heavy-tailed distributions, Section 3 presents main results and corollaries, and Sections 4–5 prove the main results and corollaries, respectively.

2. Preliminaries

Throughout the paper, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, we write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $\limsup f(x)/g(x) \le 1$ and write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$. We also write $f(x) \asymp g(x)$ if $0 < \liminf f(x)/g(x) \le \limsup f(x)/g(x) < \infty$.

For simplicity, we say that a real function $a(\cdot)$ defined on $\mathbb{R}_+$ is an auxiliary function if it satisfies $0 \le a(x) < x/2$, $a(x) \uparrow \infty$ and $a(x)/x \downarrow 0$.

As in Yang et al. (2011), for a random variable X, we introduce independent random variables $X_\vee^*$ and $X_\wedge^*$, independent of all other sources of randomness, with $X_\vee^*$ identically distributed as $X_1^* \vee X_2^*$ and with $X_\wedge^*$ identically distributed as $X_1^* \wedge X_2^*$, where X_1^* and X_2^* are two i.i.d. copies of X. Trivially, if X is distributed by F, then $X_\vee^*$ is distributed by F^2 and the tail of $X_\wedge^*$ is $\overline{F}^2$.

A distribution F on $\mathbb{R}_+$ is said to be subexponential, written as $F \in \mathcal{S}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}_+$ and $\overline{F^{2*}}(x) \sim 2\overline{F}(x)$, where F^{2*} denotes the two-fold convolution of F. More generally, a distribution F on $\mathbb{R}$ is still said to be subexponential if the distribution $F_+(x) = F(x)1_{(x\ge 0)}$ is subexponential.

A distribution F on $\mathbb{R}$ is said to be long tailed, written as $F \in \mathcal{L}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}_+$ and the relation $\overline{F}(x+y) \sim \overline{F}(x)$ holds for some (or, equivalently, for all) $y \ne 0$. For $F \in \mathcal{L}$, automatically there is an auxiliary function $a(\cdot)$ such that this relation holds uniformly for $y \in [x - a(x), x + a(x)]$; that is,

$$\lim_{x\to\infty} \sup_{x-a(x)\le y\le x+a(x)} \left| \frac{\overline{F}(x+y)}{\overline{F}(x)} - 1 \right| = 0.$$

It is well known that $\mathcal{S} \subset \mathcal{L}$; see, for example, Lemma 1.3.5(a) of Embrechts et al. (1997).

A distribution F on $\mathbb{R}$ is said to be dominatedly-varying tailed, written as $F \in \mathcal{D}$, if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}_+$ and the relation $\overline{F}(xy) = O\left(\overline{F}(x)\right)$ holds for some (or, equivalently, for all) $0 < y < 1$.

The intersection $\mathcal{L} \cap \mathcal{D}$ forms a useful subclass of $\mathcal{S}$; see Proposition 1.4.4(a) of Embrechts et al. (1997). In particular, it covers the class $\mathcal{C}$ of distributions with consistently-varying tails. By definition, for a distribution F on $\mathbb{R}$, we write $F \in \mathcal{C}$ if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}_+$ and

$$\lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

Clearly, for $F \in \mathcal{C}$ it holds for every auxiliary function $a(\cdot)$ that $\overline{F}(x + a(x)) \sim \overline{F}(x)$.

Slightly smaller than $\mathcal{C}$ is the class $\mathcal{R}$ of distributions with regularly-varying tails. By definition, for a distribution F on $\mathbb{R}$, we write $F \in \mathcal{R}_{-\alpha}$ for some $0 \le \alpha < \infty$ if $\overline{F}(x) > 0$ for all $x \in \mathbb{R}_+$ and the relation $\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x)$ holds for all $y > 0$. Denote by $\mathcal{R}$ the union of $\mathcal{R}_{-\alpha}$ over $0 \le \alpha < \infty$.

In summary, we have

$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}.$$

Moreover, a distribution F on $\mathbb{R}$ is said to be rapidly-varying tailed, written as $F \in \mathcal{R}_{-\infty}$, if the relation $\overline{F}(xy) = o\left(\overline{F}(x)\right)$ holds for all $y > 1$. This is a very broad class containing both heavy-tailed and light-tailed distributions.

For a distribution F with $\overline{F}(x) > 0$ for all $x \in \mathbb{R}_+$, its upper and lower Matuszewska indices are defined as

$$M^*(F) = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y} : y > 1\right\} \quad \text{and}$$

$$M_*(F) = \sup\left\{-\frac{\log \overline{F}^*(y)}{\log y} : y > 1\right\},$$

respectively, where $\overline{F}_*(y) = \liminf \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup \overline{F}(xy)/\overline{F}(x)$. It is clear that $F \in \mathcal{D}$ if and only if $0 \le M^*(F) < \infty$, while if $F \in \mathcal{R}_{-\alpha}$ for $0 \le \alpha \le \infty$ then $M^*(F) = M_*(F) = \alpha$.

For two independent random variables X^* and Y^* with distributions F and G, respectively, denote by $F * G$ the distribution of