Available online at www.sciencedirect.com
ScienceDirect
International Journal of Approximate Reasoning
49 (2008) 159–174
INTERNATIONAL JOURNAL OF APPROXIMATE REASONING
www.elsevier.com/locate/ijar

Financial risk measurement with imprecise probabilities

Paolo Vicig

Dipartimento di Matematica Applicata 'B. de Finetti', University of Trieste, Italy

Received 20 September 2006; received in revised form 5 June 2007; accepted 7 June 2007
Available online 21 September 2007

Abstract

Although financial risk measurement is a largely investigated research area, its relationship with imprecise probabilities has been mostly overlooked. However, risk measures can be viewed as instances of upper (or lower) previsions, thus letting us apply the theory of imprecise previsions to them. After a presentation of some well known risk measures, including Value-at-Risk or *VaR*, coherent and convex risk measures, we show how their definitions can be generalized and discuss their consistency properties. Thus, for instance, *VaR* may or may not avoid sure loss, and conditions for this can be derived. This analysis also makes us consider a very large class of imprecise previsions, which we termed convex previsions, generalizing convex risk measures. Shortfall-based measures and Dutch risk measures are also investigated. Further, conditional risks can be measured by introducing conditional convex previsions. Finally, we analyze the role in risk measurement of some important notions in the theory of imprecise probabilities, like the natural extension or the envelope theorems.

Keywords: Risk measures; Value-at-Risk (*VaR*); Coherent and convex risk measures; Imprecise previsions; Shortfall; Natural extension; Envelope theorems; Dilation

1. Introduction

Early motivations for introducing imprecise probabilities were related to problems of eliciting beliefs, especially under scarce or not quite reliable prior information, or to statistical robustness questions. Many other applications and connections with various theories have been explored subsequently. The term 'imprecise probabilities' itself includes a number of theories, like belief functions, possibility theory, and (which matters here) imprecise previsions. It is rather difficult however to find references to imprecise probabilities in the quite large literature on financial risk measurement, even though a reader acquainted with imprecise probability theory will occasionally find some analogies. The main purpose of this paper, which is a revised and extended version of [26], is to illustrate how deeply imprecise previsions may fit risk measurement problems. This analysis was performed in [19–22], where additional results and most proofs may be found; proofs are supplied here for the new material in Sections 5.4 and 5.5. The paper is primarily addressed to potential readers with some knowledge of imprecise probabilities, but little or possibly no information about risk measurement.

E-mail address: paolo.vicig@econ.units.it

0888-613X/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.ijar.2007.06.009

Therefore, Section 2 supplies some basic notions on risk measures, as they may be found in the literature, while Section 3 recalls, more concisely, notions about imprecise previsions. Some relevant concepts are defined or interpreted in different ways in the literature: we describe shortly the point of view followed in this paper in Section 2.1. The connection between imprecise previsions and risk measures is stressed in Section 4, showing that a risk measure may be viewed as an upper prevision. This lets us apply to risk measures the consistency notions developed for imprecise previsions, of coherence, of avoiding sure loss, and the somehow intermediate concept of (centered) convex prevision. We discuss various risk measures from this perspective in Section 5, while Section 6 is concerned with applying other basic concepts from imprecise prevision theory to risk measurement. The interesting but more complex issue of measuring conditional risks is presented at an introductory level in Section 5.6 and in some parts of Section 6. Notes on references may be found in the concluding Section 7.

2. Risk measurement without imprecise probabilities

A basic problem in financial risk measurement is that of stating how *risky* a given random variable X is. In practice X will be (the value in some currency of) a certain bond, a share, an index, a portfolio (i.e. a set of financial assets held by an individual or an institution) or a subportfolio (a subset of a portfolio meeting certain requirements, like the set of all options, the set of all bonds issued by a certain bank, or others), the amount of a company's insurance claims at a given date, and so on.

Although various instruments can theoretically be employed to tackle this problem, for instance loss functions, practitioners tend to favour *risk measures*, also because of their conceptual simplicity. In fact, the risk measure $\rho(X)$ for X is just a real number which should summarize the evaluation about the riskiness of X. It has a direct *operational interpretation*: when positive, it should measure the *risk capital* which the owner of X should allocate to face possible losses arising from X (cf., for instance, [9], Definition 2.2.1). Practitioners are often even more cautious: the level of the reserve funds covering risks related to portfolios of banks and other companies is determined (daily or weekly) as a multiple (for instance, two or three) of some risk measure. When negative, $\rho(X)$ represents the amount of money which could be subtracted from X, keeping the resulting random variable acceptable, or in other words *desirable*.

More generally, one might consider an arbitrary set D of random variables, and associate a real number $\rho(X)$ to each of them. The risk measure ρ is then a real function with domain D.

The outcome of each X in D will usually be determined only at a certain future time t_X (generally random, but we assume it is non-random here), while $\rho(X)$ represents an amount of money to be reserved immediately. This time gap makes the quantities $X, \rho(X)$ financially not comparable: if, say, t_X is the end of next year, we should determine to ensure comparability what is today's worth of getting X only at t_X. This worth is termed the *discounted value* of X, and may be computed multiplying X by a discounting factor $r \leqslant 1$ (quite often r is obtained from the interest rate of 'risk-free' bonds, typically government bonds). To make things simpler, we assume $r = 1$. This is not restrictive for the coming theory, and corresponds to a situation where the gap between the evaluation time and t_X is negligible, or when the discounting factor is anyway close to 1.

Clearly, the problem of choosing a risk measure is a delicate one, and it seems difficult to find proposals free of any shortcoming and criticism. We present here some solutions, among those currently most used or investigated, but will not include other important kinds of risk measures (cf. Section 7).

Probably, *Value-at-Risk* or *VaR* is nowadays the most widespread risk measure. Following [2], it is defined in this way:

Definition 1. Let X be a random variable, whose probability distribution is P. The number q is an *α-quantile* for X if

$$P(X < q) \leqslant \alpha \leqslant P(X \leqslant q). \tag{1}$$

Define then

$$q_\alpha^+(X) = \inf\{x : P(X \leqslant x) > \alpha\} \tag{2}$$

$$VaR_\alpha(X) = -q_\alpha^+(X). \tag{3}$$