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Elements for a theory of financial risks

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Abstract

Estimating and controlling large risks has become one of the main concerns of financial institutions. This requires the development of adequate statistical models and theoretical tools (which go beyond the traditional theories based on Gaussian statistics), and their practical implementation. Here we describe two interrelated aspects of this program: we first give a brief survey of the peculiar statistical properties of the empirical price fluctuations. We then review how an option pricing theory consistent with these statistical features can be constructed, and compared with real market prices for options.

Keywords: Financial analysis; Scaling; Power-law correlations

1. Introduction

The efficiency of the theoretical tools devised to address the problems of risk control, portfolio selection and derivative pricing strongly depends on the adequacy of the stochastic model chosen to describe the market fluctuations. Historically, the idea that price changes could be modelled as a Brownian motion dates back to Bachelier [1]. This hypothesis, or some of its variants (such as the geometrical Brownian motion, where the log of the price is a Brownian motion) is at the root of most of the modern results of mathematical finance, with Markowitz portfolio analysis, the capital asset pricing model (CAPM) [2] and the Black–Scholes formula [3] standing out as paradigms. The reason for success is mainly due to the impressive mathematical and probabilistic apparatus available to deal with Brownian motion problems, in particular

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Ito's stochastic calculus. An important justification of the Brownian motion description lies in the central limit theorem (CLT), stating that the sum of N identically distributed, weakly dependent random changes is, for large N, a Gaussian variable. In physics or in finance, the number of elementary changes observed during a time interval t is given by $N = t/\tau^*$, where τ^* is an elementary correlation time, below which changes of velocity (for the case of a Brownian particle) or changes of 'trend' (in the case of the stock prices) cannot occur. The use of the CLT to substantiate the use of Gaussian statistics in any case requires that $t \gg \tau^*$. In financial markets, τ^* turns out to be of the order of several minutes, which is not that small compared to the relevant time scales (days), in particular when one has to worry about the *tails* of the distribution (corresponding to large shocks) which sometimes disappear only very slowly. The fact that the tails of the distribution of returns are much 'fatter' than predicted by the Gaussian is well known, in particular since the seminal work of Mandelbrot [4,5], where the idea that price changes are still independent, but distributed according to a Lévy stable law was first proposed. This model however fails in two respects. First, the tails of the distribution of returns is now much *overestimated*; in particular, the variance appears to be well defined in most financial markets, while it is infinite for Lévy distributions. Second, and perhaps more importantly in view of its application to option markets, the amplitude of the fluctuations (measured, say, as the local variance) appears to be itself a randomly fluctuating variable with rather long-range correlations.

The aim of these lectures is to provide a short survey of the most prominent statistical properties of the fluctuations of rather liquid markets, which are characterized by what one could call 'moderate' fluctuations (for a review, see e.g. Refs. [6–8]). Less liquid markets sometimes behave rather differently and more 'extreme' fluctuations can be observed. We shall then present a theory for option pricing and hedging in the general case where the underlying stock fluctuations are not Gaussian. In this case, perfect hedging is in general impossible, but optimal strategies can be found (analytically or numerically) and the associated residual risk can be estimated. We show that the volatility 'smile' observed on option markets can be understood using a cumulant expansion, and discuss the idea of an implied 'kurtosis', which is (on liquid markets) very close to the actual (maturity dependent) kurtosis of the historical data.

2. A short survey of empirical data

2.1. Linear correlations

We shall denote in the following the value of the stock (or any other asset) at time t as $x(t)$, and the variation of the stock on a given time interval τ as $\delta_\tau x(t) = x(t+\tau) - x(t)$. The time delay τ can be as small as a few seconds in actively traded markets. However, on these short time scales, the fluctuations cannot be considered to be independent. For