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Approximating the distributions of estimators of financial risk under an asymmetric Laplace law

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Abstract

Explicit expressions are derived for parametric and nonparametric estimators (NPEs) of two measures of financial risk, value-at-risk (VaR) and conditional value-at-risk (CVaR), under random sampling from the asymmetric Laplace (AL) distribution. Asymptotic distributions are established under very general conditions. Finite sample distributions are investigated by means of saddlepoint approximations. The latter are highly computationally intensive, requiring novel approaches to approximate moments and special functions that arise in the evaluation of the moment generating functions. Plots of the resulting density functions shed new light on the quality of the estimators. Calculations for CVaR reveal that the NPE enjoys greater asymptotic efficiency relative to the parametric estimator than is the case for VaR. An application of the methodology in modeling currency exchange rates suggests that the AL distribution is successful in capturing the peakedness, leptokurticity, and skewness, inherent in such data. A demonstrated superiority in the resulting parametric-based inferences delivers an important message to the practitioner.

Keywords: Value-at-risk; Expected shortfall; Saddlepoint approximation; Order statistic; Exchange rate

1. Introduction

Increasing demand by regulatory bodies, such as the Basel Committee on Banking supervision, to implement methodically sound credit risk assessment practices, has fostered much recent research into measures of financial risk. Two common such measures are *value-at-risk* (VaR) and *conditional value-at-risk* (CVaR). Our choice of name for the latter is drawn from Rockafellar and Uryasev (2000), but synonyms for it also in common usage include: “expected shortfall” (Acerbi and Tasche, 2002), and “tail-conditional expectation” (Artzner et al., 1999). If Y denotes the random variable that gives rise to the risk being measured, VaR and CVaR attempt to quantify the seriousness of possible losses associated with Y by providing measures of location out on its tails. Loosely speaking, if Y represents “losses” and $0 < \alpha < 1$, the VaR of Y at probability level α, $\text{VaR}_\alpha(Y)$, is a lower bound on the worst $(1-\alpha)100\%$ losses; while $\text{CVaR}_\alpha(Y)$ is the average of these worst $(1-\alpha)100\%$ losses.

In the insurance industry, for example, accurate estimation of these risk measures for high quantiles like $\alpha = 0.99$ of the insured values is of paramount importance in increasingly uncertain times. In financial portfolio management,

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risk measures can be used as optimality criteria in determining an "optimal" portfolio, but are more often employed as constraints in conjunction with optimality criteria such as maximization of expected profit (Uryasev, 2000). In this context the conceptually simpler VaR quickly became popular, but suffered from conceptual and numerical problems (Artzner et al., 1997). CVaR arose out of attempts to correct these shortcomings (Artzner et al., 1997, 1999). Recent work by Rockafellar et al. (2006) solidifies these ideas by introducing the concept of a generalized deviation measure.

In order to proceed further, we introduce some notation and give precise definitions. Let Y be a continuous real-valued random variable defined on some probability space $(\Omega, \mathscr{A}, P)$, with cumulative distribution function (cdf) $F(\cdot)$ and probability density function (pdf) $f(\cdot)$. (In this paper we consider only the case of continuous Y; for definitions in the general case we refer the interested reader to Rockafellar and Uryasev (2002).) The quantities μ and σ^2 will denote the mean and variance of Y, respectively, and both are assumed to be finite. With the understanding that Y represents loss, the VaR of Y at probability level α is defined to be the αth quantile of Y.

Definition 1 (*VaR*).

$$\mathrm{VaR}_\alpha(Y) \equiv \xi_\alpha(Y) = F^{-1}(\alpha). \tag{1}$$

The CVaR of Y at probability level α is the mean of the random variable that results by truncating Y at ξ_α and discarding its lower tail.

Definition 2 (*CVaR*).

$$\mathrm{CVaR}_\alpha(Y) \equiv \mu_\alpha(Y) = \mathbb{E}\,(Y|Y \geqslant \xi_\alpha) = \frac{1}{1-\alpha}\int_{\xi_\alpha}^{\infty} y f(y)\,\mathrm{d}y. \tag{2}$$

When no ambiguity arises we write simply ξ_α and μ_α. Typical values for α are in the range $0.90 \leqslant \alpha \leqslant 0.99$. If negative values of Y represent losses, the CVaR of Y at probability level α is defined to be $\mu_\alpha(-Y)$. An equivalent definition of CVaR in terms of the quantile function of Y is

$$\mu_\alpha(Y) = \frac{1}{1-\alpha}\int_{\alpha}^{1} F^{-1}(u)\,\mathrm{d}u.$$

There has been earlier interest in estimating population parameters akin to CVaR in other fields besides finance. In reliability and survival analysis, for example, the *mean residual life* at time t is defined to be $\mathbb{E}(Y - t|Y > t)$, which can be equivalently expressed in terms of CVaR as $\mu_{F^{-1}(t)} - t$. The quantity $\left(\mu_\alpha - \mu\right)/\sigma$ has appeared in the genetics literature since the 1960s, where it has become known as the *selection differential* (Nagaraja, 1988). If the top $(1-\alpha)100\%$ of a finite population is selected for interbreeding purposes, the selection differential represents the standardized improvement upon selection.

The primary intent of this paper is to compare parametric and nonparametric estimators (NPEs) of VaR and CVaR, both in finite samples and asymptotically. To this end, we describe in Section 2 a flexible parametric family that can be used for modeling certain types of financial data, the asymmetric Laplace (AL) distribution, and give explicit expressions for estimators of VaR and CVaR in that context. The asymptotic distribution of the bivariate (VaR,CVaR) estimator is established, and its finite sample properties are explored by saddlepoint approximating the individual distributions. NPEs are similarly investigated in Section 3. Comparisons between the parametric and NPEs are made in Section 4, in terms of asymptotic relative efficiencies (AREs) for large samples, and via saddlepoint approximations of density functions in finite samples. We conclude in Section 5 with an application of the methodology in modeling daily US dollar (USD)/EU euro (EUR)exchange rates.

2. Parametric estimation

This section discusses maximum likelihood estimation of VaR and CVaR under random sampling from an AL law. The joint asymptotic distribution of the two estimators is derived. Using saddlepoint methods, accurate approximations to the finite sample distributions are computed.