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Comonotonicity, efficient risk-sharing and equilibria in markets with short-selling for concave law-invariant utilities☆

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ABSTRACT

In finite markets with short-selling, conditions on agents' utilities insuring the existence of efficient allocations and equilibria are by now well understood. In infinite markets, a standard assumption is to assume that the individually rational utility set is compact. Its drawback is that one does not know whether this assumption holds except for very few examples as strictly risk averse expected utility maximizers with same priors. The contribution of the paper is to show that existence holds for the class of strictly concave second order stochastic dominance preserving utilities. In our setting, it coincides with the class of strictly concave law-invariant utilities. A key tool of the analysis is the domination result of Lansberger and Meilijson that states that attention may be restricted to comonotone allocations of aggregate risk. Efficient allocations are characterized as the solutions of utility weighted problems with weights expressed in terms of the asymptotic slopes of the restrictions of agents' utilities to constants. The class of utilities which is used is shown to be stable under aggregation.

1. Introduction

The problem of the existence of equilibria in finite markets with short-selling has first been considered in the early seventies by Grandmont (1977), Hart (1974) and Green (1973) in the context of temporary equilibrium models and assets equilibrium models. It was later reconsidered by a number of authors (for a review of the subject in finite markets, see Allouch et al., 2002; Dana et al., 1999; Page, 1996). Three sets of conditions were given for existence of an equilibrium:

- the assumption of existence of a no-arbitrage price, a price at which no investor could make costless unbounded utility nondecreasing purchases (see for example Grandmont, 1977; Hammond, 1983; Page, 1987; Werner, 1987) or equivalently under standard conditions on utilities, that aggregate demand exists at some price,
- the no unbounded utility arbitrage condition, a condition of absence of collective arbitrage, which requires that investors do not engage in mutually compatible, utility nondecreasing trades (see for example, Hart, 1974; Page, 1987; Nielsen, 1989),
- and finally, the assumption that the individually rational utility set is compact (see for example, Dana et al., 1999; Nielsen, 1989; Page and Wooders, 1996). Under suitable assumptions, these conditions were shown to be equivalent (see Dana et al., 1999; Page and Wooders, 1996).

While the problem of existence of an Arrow–Debreu equilibrium in infinite economies with consumption sets bounded below was considered as well understood at the end of the eighties (see Aliprantis et al., 1989; Mas-Collel and Zame, 2001), a number of papers discussed the difficulties raised by the issue of shortselling: Cheng (1991), Brown and Werner (1993), Dana and Le Van (1995), Dana and Le Van (2000), Dana et al. (1997) and Aliprantis et al. (1998), this list not being exhaustive. The finite dimension assumptions where shown not to be equivalent, the assumption of absence of free lunch or of absence of collective arbitrage too weak. The standard assumption has been to assume that the individually rational utility set is compact with the drawback that it is not known whether it is fulfilled except for very few examples (models with mean variance utilities or strictly risk averse expected utilities). For proving existence of equilibrium in infinite dimension economies with consumption sets unbounded below, most papers have used the topological version of Negishi's approach. Dana and Le Van (1995) and Dana and Le Van (2000) have used utility weights e and the excess utility correspondence. Their paper however relies on the assumption that the individually rational utility set is compact.

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More recently, there has been renewed interest for the problem of existence and characterization of efficient allocations in markets with short-selling, in the mathematical finance literature. Indeed, for the last 10 years, the problem of quantifying the risk of a financial position has been very popular in finance (see Föllmer and Schied, 2004, for an overview) and has led to the concept of *convex measure of risk*. Risk sharing of an aggregate capital between different units or different investors or of the risk of a bank between its subsidiaries, led to problems of efficiency with shortselling that have been mainly discussed in infinite dimension (see for example, Barrieu and El Karoui, 2002; Filipovic and Swidland, 2008; Jouini et al., 2008). All of these papers have all considered law-invariant convex measures of risk.

To show existence of efficient allocations for law-invariant convex measures of risk, Filipovic and Swidland (2008) and Jouini et al. (2008) have both used *the domination result* of Landsberger and Meilijson (1994) that any allocation of an aggregate risk is dominated for second order stochastic dominance by a comonotone allocation. Comonotone allocations are allocations having the property that agents' wealths are non decreasing functions of aggregate wealth that add up to identity. They are said to fulfill a *mutuality principle*. Moreover, the wealths of any pair of agents are positively correlated. Since the early work of Borch (1962), Arrow (1963) and Wilson (1968), they have played an important role in the theory of risk sharing between strictly concave expected utility maximizers, the efficient allocations of risk being comonotone. When utilities are second order stochastic dominance preserving, from the domination result, for efficiency issues, attention may be restricted to comonotone allocations. As comonotone allocations are almost compact, with mild continuity assumptions on utilities, the individually rational utility set is compact. When the state space is non atomic and the utilities are *concave*, the hypothesis that utilities are second order stochastic dominance preserving is equivalent to their law invariance. By definition, law invariant utility functions only depend on the distributions of wealths and include many standard utilities, the expected utility, the rank dependent expected utility, the prospect utility, Green and Jullien's utility (see below), the opposite of a number of very well-known risk-measures used in finance as entropy or averagevar. They have been very popular in the decision theoretic literature of the eighties. However not all of them are concave. For example risk averse expected utilities are law invariant and concave while risk taker expected utilities are law invariant and convex.

The first aim of the paper is to show existence of efficient allocations and equilibria for markets with short-selling for concave second order stochastic dominance preserving utilities that fulfill some mild continuity properties and are strictly concave for most of the agents. In view of unifying models used in economics and in finance, an $l+m$ agents exchange economy is considered, the first l agents having monetary utilities (adding t units of cash to a position increases the utility of t) and the last m agents having law invariant strictly concave utilities. State contingent claims are assumed to be in L^∞, a choice that may seem odd given current financial markets and the horrors of $(L^\infty)'$. As the utilities that are considered in the paper have supergradients in L^1_+, attention may be restricted to countably additive prices (or prices in L^1_+) and values of contingent claims are integrals with respect to pricing densities.

Since utility functions are concave, the utility weight version of Negishi's method may be used to show existence of efficient allocations and equilibria. Efficient allocations are characterized as the solutions of utility weighted problems for weights expressed in terms of the asymptotic slopes of the restrictions to constants of the agents' utilities. The same utility weights characterize the efficient sharings of a fixed amount of a non random wealth between $l+m$ agents having as utilities on the reals the restrictions to reals of agents' utilities. The efficient utility weights are therefore defined as the solutions of a set of equalities (monetary agents have same weight, otherwise, they would exchange cash so as to increase aggregate utility) and strict inequalities expressed in terms of the asymptotic slopes of the restrictions to constants of the agents' utilities.

The second aim of the paper is to show that the class of utilities being studied is stable by aggregation. The aggregation problem is by no mean an easy problem. The strictly risk averse RDU class is not stable by aggregation while the class of Choquet integrals with respect to a convex distortion is stable. It is shown that the monetary agents have a representative agent with a monetary law invariant utility (the sup-convolution of the monetary agents' utilities). Strictly concave agents also have a representative agent but it depends on the efficient allocation considered (or on the set of utility weights characterizing the efficient allocation). At any efficient allocation, the representative agent of the whole economy has a law invariant strictly concave utility. Finally at any efficient allocations, the wealths of the strictly concave agents and the aggregate monetary wealth are comonotone. The anticomonotonicity of prices and aggregate risk is known to hold in some cases. The generality of the result remains an open question.

The paper is organized as follows. In Section 2, the model is presented and l some properties of law invariant, concave, norm continuous utilities recalled. Using the domination result of Lansberger and Meilijson, the utility set is shown to be closed and the individually rational utility set is compact. Section 3 is devoted to the characterization of efficient allocations as solutions of utility weighted problems. A monetary representative agent independent of utility weights is introduced. Section 4 is devoted to existence of equilibria and to some of its qualitative and aggregation properties. Appendix A contains the proof of the two main results of the paper.

2. The domination result for law invariant concave utilities

2.1. The model

We consider a standard Arrow–Debreu one good exchange economy under uncertainty with $l+m$ agents. Agents trade the set of state-contingent claims and have homogeneous beliefs about states of the world. Given as primitive is a non-atomic probability space $(\Omega, \mathcal{B}, P)$, hence it supports a random variable U uniformly distributed on [0, 1]. Contingent claims are identified to elements of $L^\infty(\Omega, \mathbb{R})$ that we now on write L^∞. Agents are described by their endowments $W_i \in L^\infty$, $i=1, \ldots, l+m$ and their utilities. Let $W := \sum_i W_i$ be the aggregate endowment with distribution function F_W assumed to be continuous. Agents' utilities, $u_i : L^\infty \to \mathbb{R}$ are concave, monotone, law invariant (two random variables with same probability law, have same utility), continuous in the norm topology of Ł$^\infty$. We also assume that the utility of some agent fulfills the following continuity assumption that insures that the superdifferential of the utility is in Ł^1 (see Proposition 2 below).

H $X_n \uparrow X$: a.e. implies $u(X_n) \uparrow u(X)$.

A utility is monetary if it is monotone and fulfills

$$u_i(X+t) = u_i(X) + t \text{ for any } t \in \mathbb{R},$$

In other words, if the risk-free amount $t \in \mathbb{R}$ is added to X, then the utility increases of t. The opposite of a concave monotone, monetary utility is called a *convex measure of risk*. Numerous examples of concave, monotone, law invariant monetary utilities may be found in Jouini et al. (2008) and in Föllmer and Schied (2004).