Journal of Mathematical Economics 53 (2014) 101–105

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Efficient allocations and equilibria with short-selling and incomplete preferences

R.A. Dana [a,*], C. Le Van [b]

[a] IPAG Business School, CEREMADE, Université Paris-Dauphine, Pl. du maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France
[b] IPAG Business School, CNRS, PSE, VCREME, CES, 106-112 Bd de l'Hôpital, 75647 Paris Cedex 13, France

ARTICLE INFO

Article history:
Received 9 August 2013
Received in revised form
24 May 2014
Accepted 9 June 2014
Available online 17 June 2014

Keywords:
Uncertainty
Risk
Risk adjusted prior
No arbitrage
Equilibrium with short-selling
Incomplete preferences

ABSTRACT

This paper reconsiders the theory of existence of efficient allocations and equilibria when consumption sets are unbounded below under the assumption that agents have incomplete preferences. Our model is motivated by an example in the theory of assets with short-selling where there is risk and ambiguity. Agents have Bewley's incomplete preferences. As an inertia principle is assumed in markets, equilibria are individually rational. It is shown that a necessary and sufficient condition for the existence of an individually rational efficient allocation or of an equilibrium is that the relative interiors of the risk adjusted sets of probabilities intersect. The more risk averse, the more ambiguity averse the agents, the more likely is an equilibrium to exist. The paper then turns to incomplete preferences represented by a family of concave utility functions. Several definitions of efficiency and of equilibrium with inertia are considered. Sufficient conditions and necessary and sufficient conditions are given for the existence of efficient allocations and equilibria with inertia.

1. Introduction

The issue of existence of an equilibrium for finite markets with short-selling is an old problem first considered in the early seventies by Grandmont (1970), Hart (1974) and Green (1973) and reconsidered later by Hammond (1983) and Page (1987, 1996). In these early papers, investors were assumed to hold a single probabilistic belief (homogeneous or heterogeneous) and be risk averse von Neumann–Morgenstern (vNM) utility maximizers. Two sufficient conditions for the existence of an equilibrium were given:

1. a condition which expresses that investors are sufficiently similar in their beliefs and risk aversions so that there exists a non empty set of prices (*the no-arbitrage prices*) for which no agent can make costless unbounded utility nondecreasing purchases,
2. a collective absence of arbitrage condition, which requires that investors do not engage in mutually compatible, utility nondecreasing trades.

These conditions have been generalized to abstract economies and are known as the *existence of no-arbitrage prices* condition (see Werner, 1987) and the *no unbounded utility arbitrage* condition (NUBA) (see Page, 1996). They were shown to be equivalent under adequate hypotheses. Other sufficient conditions were given. For a review of the subject in finite dimension, see Allouch et al. (2002), Dana et al. (1999) and Page (1987, 1996). All this trend of literature assumes complete preferences.

This paper extends the previous theory to incomplete preferences represented by families of concave utility functions. Such incomplete preferences include Bewley's (2002) and Rigotti and Shannon's (2005) incomplete preferences. Under this representation assumption, it is easy to define and characterize the concepts of no-arbitrage prices and no unbounded utility arbitrage. Weak and strong concepts of efficiency and equilibria are defined. As in the case of complete preferences, the existence of a no-arbitrage price is shown to be equivalent to NUBA and to be a sufficient condition for the existence of weakly efficient allocations and weak equilibria. Under further adequate assumptions (for example strict concavity and monotonicity of utilities), it is shown to be necessary and sufficient for the existence of efficient allocations and equilibria.

As Page (1996) and Werner (1987), the paper is motivated by an example in the theory of assets with short-selling. Agents are assumed not to have enough information to quantify uncertainty by a single probability; hence each agent has a set of priors. Agents are further assumed to have risk averse Bewley's (2002) (or Rigotti and Shannon, 2005) incomplete preferences. Under standard

* Corresponding author.
E-mail addresses: dana@ceremade.dauphine.fr (R.A. Dana), levan@univ-paris1.fr (C. Le Van).

http://dx.doi.org/10.1016/j.jmateco.2014.06.003
0304-4068/

conditions on utility indices (strict concavity and increasingness) and sets of priors (convexity and compactness), it is shown that a necessary and sufficient condition for the existence of an individually rational efficient allocation or equilibrium is that the relative interiors of the agents' sets of risk adjusted probabilities intersect or that agents do not engage in mutually compatible trades that have non negative expectations with respect to their risk adjusted probabilities. The first condition generalizes the overlapping expectations given by Grandmont (1970), Green (1973) and Hammond (1983) for the case of single belief. The second generalizes the NUBA type of condition given by Hart (1974). In a vNM framework, Hart (1974) further shows that the more risk averse the agents, the more likely is an equilibrium to exist. In Bewley's setting, it is shown that the more risk averse, the more ambiguity averse the agents, the more likely is an equilibrium to exist.

The example is also shown to have an interesting feature: the arbitrage concepts coincide with those of a Gilboa–Schmeidler's agent with the same sets of priors and utility indices. Hence the condition that the relative interiors of the sets of risk adjusted probabilities intersect is also necessary and sufficient for the existence of an equilibrium in a model with Gilboa–Schmeidler's preferences (with the same sets of priors and utility indices). This may seem at odds with the literature on efficiency or equilibria with Bewley's incomplete preferences (see Bewley, 2002, Rigotti and Shannon, 2005 or Dana and Riedel, 2013) which highlights the differences with using maxmin complete preferences. However it is not, as a different issue is addressed, that of existence of equilibrium with short-selling.

The paper is organized as follows: Section 2 deals with the example and provides an existence theorem while Section 3 deals with its generalization.

2. An example

2.1. Bewley's preferences

We consider a standard Arrow–Debreu model of complete contingent security markets with short-selling. There are two dates, 0 and 1. At date 0, there is uncertainty about which state s from a state space $\Omega = \{1, \dots, k\}$ will occur at date 1. At date 0, agents who are uncertain about their future endowments trade contingent claims for date 1. The space of contingent claims is the set of random variables from $\Omega \to \mathbb{R}$. The random variable X which equals x_1 in state 1, x_2 in state 2 and x_k in state k, is identified with the vector in $X \in \mathbb{R}^k$, $X = (x_1, \dots, x_k)$. Let $\triangle = \{\pi \in \mathbb{R}^k_+ : \sum_{s=1}^k \pi_s = 1\}$ be the probability simplex in $\mathbb{R}^k$. Let $\operatorname{int}\triangle = \{\pi \in \triangle, \pi_s > 0, \ \forall \ s\}$. For $A \subseteq \triangle$, $\operatorname{int} A = \{p \in A \mid \exists$ a ball $B(p, \varepsilon)$ s.t. $B(p, \varepsilon) \cap \operatorname{int}\triangle \subseteq A\}$. For a given $\pi \in \triangle$, we denote by $E_\pi(X) := \sum_{l=1}^k \pi_l x_l$ the expectation of X. Finally, for a given price $p \in \mathbb{R}^k$, $p \cdot X := \sum_{l=1}^k p_l x_l$, the price of X.

There are m agents indexed by $i = 1, \dots, m$. Agent i has an endowment $E^i \in \mathbb{R}^k$ of contingent claims. Let $(E^i)_{i=1}^m$ be the m-tuple of endowments and $E = \sum_{i=1}^m E^i$ be the aggregate endowment. We assume that agent i has a convex compact set of priors $P^i \subseteq \operatorname{int}\triangle$ and an incomplete Bewley's preference relation $\succeq$ over $\mathbb{R}^k$ defined by

$$X \succeq^i Y \iff E_\pi(u^i(X)) \geq E_\pi(u^i(Y)), \quad \forall \pi \in P^i \tag{1}$$

where $u^i : \mathbb{R} \to \mathbb{R}$ is a strictly concave, increasing differentiable utility index fulfilling $u^i(0) = 0$. The associated strict preference is $X \succ^i Y$ if $X \succeq^i Y$ and $E_\pi(u^i(X)) > E_\pi(u^i(Y))$ for some $\pi \in P^i$.

2.2. Individual and collective absence of arbitrage

In this subsection, we define and characterize the useful vectors of a Bewley's preference relation of type (1). They are the directions such that trading at any positive scale makes the agent better off. The main result is that they coincide with those of a Gilboa–Schmeidler's utility defined by (u, P) (see (2) below).

We then recall the concepts of no-arbitrage prices and of collective absence of arbitrage (NUBA). As these concepts only depend on useful vectors, we obtain that two economies with Bewley's preferences or Gilboa–Schmeidler's utilities with the same indices and sets of priors have the same sets of no-arbitrage prices and the same NUBA condition.

2.2.1. Useful vectors

To simplify notations, in this subsection, the agent's index is omitted. We consider an agent described by a pair (u, P) of a utility index and a set of priors. For $\pi \in P$, let

$$\widehat{P}_\pi(X) = \{Y \in \mathbb{R}^k \mid E_\pi(u(Y)) \geq E_\pi(u(X))\}$$

be the set of contingent claims preferred to X for the utility $E_\pi(u(.))$ and $R_\pi(X) = \{W \in \mathbb{R}^k \mid E_\pi(u(X + \lambda W)) \geq u(X), \ \forall \ \lambda \in \mathbb{R}_+\}$ be its asymptotic cone. As $E_\pi(u(.))$ is concave, $R_\pi(X)$ is independent of X and denoted R_π. Taking $X = 0$, we obtain

$$R_\pi = \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq u(0) = 0, \ \forall \ \lambda \in \mathbb{R}_+\}.$$

Let

$$\widehat{P}(X) = \{Y \in \mathbb{R}^k \mid E_\pi(u(Y)) \geq E_\pi(u(X)), \ \forall \ \pi \in P\}$$

be the set of contingent claims preferred to X for the Bewley's preference defined by (1) and $R(X)$ be its asymptotic cone. From Rockafellar's (1970, Corollary 8.3.3), $R(X) = \cap_{\pi \in P} R_\pi(X) = \cap_{\pi \in P} R_\pi$. Hence it is independent of X and denoted R:

$$R = \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq 0, \ \forall \ \lambda \in \mathbb{R}_+, \ \pi \in P\}$$

and is called the set of *useful vectors* for $\succeq$. For a given pair (u, P), let

$$V(X) = \min_{\pi \in P} E_\pi(u(X)) \tag{2}$$

be Gilboa–Schmeidler's utility.

$$\{W \in \mathbb{R}^k \mid V(\lambda W) \geq 0, \ \forall \ \lambda \in \mathbb{R}_+\}$$
$$= \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq 0, \ \forall \ \lambda \in \mathbb{R}_+, \ \pi \in P\}.$$

Hence it coincides with R. Furthermore given $C \in \mathbb{R}^k$ a point, the C-reference dependent ambiguity averse (RAA) utility, axiomatized by Mihm (2010) is defined by

$$V_C(X) = \min_{\pi \in P}[E_\pi(u(X)) - E_\pi(u(C))]. \tag{3}$$

This is a concave variational utility; hence the set of useful vectors is independent of X. As $V_C(C) = 0$, $R_{V_C} = \{W \in \mathbb{R}^k \mid V_C(C + \lambda W) \geq 0, \ \forall \ \lambda \in \mathbb{R}_+\}$. Equivalently

$$R_{V_C} = \{W \in \mathbb{R}^k \mid E_\pi(u(C + \lambda W)) \geq E_\pi(u(C)), \ \forall \ \lambda \in \mathbb{R}_+, \pi \in P\} = R.$$

The previous discussion is summarized in the following lemma:

Lemma 1. *The set of useful vectors for Bewley's preference of type (1) coincides with the set of useful vectors for Gilboa–Schmeidler's preference (2) or of an RAA utility (3) for any point $C \in \mathbb{R}^k$.*

Let us recall a characterization of useful vectors proven in Dana and Le Van (2010). To this end, let

$$\widetilde{P} = \left\{ p \in \triangle \mid \exists \ \pi \in P, \ Z \in \mathbb{R}^k \text{ s.t. } p_s = \frac{\pi_s u'(z_s)}{E_\pi(u'(Z))}, \ \forall \ s = 1, \dots, k \right\} \tag{4}$$

be the set of risk adjusted probabilities. We have (see Dana and Le Van, 2010) the following lemma.