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An equilibrium existence result with short selling

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Abstract

We provide a proof of the existence of a quasi-equilibrium in an exchange economy with short selling when the preference relations of the investors are represented by a partial preorder. This setting encompasses the case of preference relations derived from a utility function. A second purpose of this paper is to compare the compactness condition used in this existence result with some compactness concepts found in the literature. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The main contribution of this paper is a proof of the existence of a quasi-equilibrium in an exchange economy with short selling when the preference relations of the investors are represented by a partial preorder. This setting encompasses the case of preference relations derived from a utility function. In a securities trading model, the possibility of short sales implies that the investment portfolios are not bounded from below. Thus, the standard assumptions made to ensure the existence of market clearing prices may not hold. In fact, an equilibrium may not exist. The equilibrium existence problem in such models has been studied in an asset market setting by Hart (1974), Hammond (1983), Page (1987) and in a general equilibrium setting by Werner (1987), Nielsen (1989), Page and Wooders (1996) and Dana et al. (1999), among others. Unlike our model, these authors rely on utility-representable preferences.

In an exchange economy model with finitely many securities, Hart (1974) considers a two-period economy in which trading in securities takes place in the first period and security

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returns are determined in the second period. Investors are assumed to be exclusively interested in their wealth in the second period. Each investor has a von Neumann–Morgenstern utility function u_i over portfolio return. The utility of portfolio x is then the expected utility of its return, $V_i(x) = Eu_i(r \cdot x)$, where r is the random vector of returns per unit of assets. So calculated, V_i can be thought of as the utility function of consumer i in a one-period exchange economy $\mathcal{E} = (X_i, V_i, e_i)$. This is the approach taken by Werner (1987) among others.

However, the expected utility is neither the only nor the most adequate formalization for investors' preferences. The two following examples illustrate the usefulness of our partial preorder approach. First, typically, the return of a portfolio is described by a point in a payoff space. This is a space of real-valued random variables on some underlying probability measure space (Ω, Σ, P), such as, $L_p(\Omega, \Sigma, P)$ for $1 \leq p \leq +\infty$. In many instances, the natural partial order of L_p can represent investors' preferences on portfolios. This is the point of view of Aliprantis et al. (1998). A second interpretation of the preorder consists of an ordering of the real-valued bounded random variable space $L_\infty(\Omega, \Sigma, P)$ with second order stochastic dominance i.e., $Y \preceq_{\mathrm{ssd}} X$, if $\int_{-\infty}^{x} P(X \leq t)\mathrm{d}t \leq \int_{-\infty}^{x} P(Y \leq t)\mathrm{d}t$, $\forall x \in R$. This can be interpreted as X being (weakly) less risky than Y, in accordance with Rothshild and Stiglitz (1970) definition in the particular case where X and Y have the same mean.

A first aim of the present paper is to extend several equilibrium existence results to the case where the preference relations of the investors are represented by a partial preorder. This case encompasses the earlier examples as well as the case of preference relations derived from a utility function. As in the previous literature, we introduce a compactness condition. The compactness with partial preorder (CPP) condition eliminates the problem introduced by unboundedness of consumption sets and, therefore, implies the existence of a competitive equilibrium. Following Gale and Mas-Colell (1975) and Shafer (1976), the idea of the proof consists of considering a simultaneous optimization problem with a finite number of $m + 1$ agents, where m is the number of investors, plus one market agent, who determines the price system.

A second aim of this paper is to compare, within the framework adopted, the CPP condition used in the existence result with other compactness concepts found in the literature. Our result is implied by Hart (1974) and Page (1987), but is equivalent to Dana et al. (1999) in the case of utility-representable preferences.

The paper is organized as follows. In Section 2, we describe the model. In Section 3, we introduce the CPP condition and prove the main existence result (Theorem 3.1). In Section 4, we extend some compactness notions used in the literature to the partially preordered case. In Section 5, we restrict ourselves to the case of complete preordering preferences represented by utility functions and discuss the relationship between the CPP condition and the compactness of the utility set. Finally, the Appendix is devoted to some of the proofs.

2. The model

We consider an exchange economy $\mathcal{E} = (X_i, \preceq_i, e_i)_{i=1,\dots,m}$ with positive finite numbers ℓ of assets and m of investors. For every $i = 1, \dots, m$, $X_i \subset R^\ell$ is the choice set of the