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Extensions of Cournot duopoly: An applied mathematical view[☆]

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ABSTRACT

The aim of the present paper is to analyze, from a dynamical system's point of view, possible extensions of the classical economic situation of the Cournot duopoly where two firms producing an identical good compete with full information for the market making decisions in each step of the process in terms of what the rival firm did in the previous stage. We generalize this situation to a model with n firms and we discuss different variations of this model in terms of the amount of information available to each firm from the point of view of the dynamical complexity. Finally, we introduce a new model where the firms compete "à la Cournot" in a local way and the information level of each firm depends on the number of firms and on the size of the so-called influence neighboring set.

1. First extension: From duopoly to oligopoly

The Cournot duopoly was introduced by Agustin Cournot [1] who is considered one of the forerunners of modern microeconomics. The process consists of two firms which produce an identical good and which compete for the market. In each step of the process the firms decide the amount of product which is to be introduced in the market and for making this decision both firms know the amount of product introduced in the market in the previous step by the rival firm. This economic process is simulated by the following two-dimensional discrete dynamical system

$$F(x, y) = (g(y), f(x)) \qquad (1)$$

where f, g are continuous self-maps defined on a compact interval which can be considered, without loss of generality, by the normalization $[0, 1]$. The maps f and g are called *reaction functions* and determine the decisions taken by the firms.

Note that if firm A put on the market at the beginning of the game α_0 product and firm B put β_0, in the next step of the game firm A will produce $g(\beta_0)$, i.e., an amount of product which directly depends on the production level of firm B in the previous step, on the other hand firm B will produce $f(\alpha_0)$ and so on. Therefore, all the processes are governed by the dynamics of the discrete system (1) which strongly depends on the dynamics of the one-dimensional interval maps f and g.

Duopoly is an intermediate situation between monopoly and perfect competition, and analytically is a more complicated case. The reason for this is that an oligopolist must consider not only the behaviour of the costumers, but also those of the competitors and their reactions. This model has been studied in the literature from different points of view, see for instance [2–7] or [8].

Given a model, economists need to make predictions on the asymptotic behaviour of the system, i.e., how the model behaves in the future. To do this it is essential to have a tool which allows us to measure the dynamical complexity of

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the model. It is known that literature is plenty of examples of seemingly simple systems which have a very complicated dynamics, i.e., chaotic behaviour, where is not possible to deduce reliable information on the future dynamics (see for instance papers [8–10] where some models have analyzed from a numerical point of view).

Specifically in [11], a topological characterization for their dynamical complexity of (1) was given. This result is exactly the generalization for two-dimensional maps of the form $(x, y) \to (g(y), f(x))$ on the one hand of the one-dimensional Misiurewicz's theorem (see [12], (1) $\Leftrightarrow$ (2)) and on the other hand of certain results proved by Sharkovskiĭ in the sixties (see [13], (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4)). By $h(\cdot)$, UR$(\cdot)$, Rec$(\cdot)$ and AP$(\cdot)$ we respectively denote the *topological entropy* and the sets of *uniformly recurrent*, *recurrent* and *almost periodic points* (for definitions see [14]).

Theorem 1 (*Misiurewicz, Sharkovskiĭ*)**.** *Let $\phi : [0, 1] \to [0, 1]$ be a continuous map. The following properties are equivalent:*

(1) $h(\phi) = 0$,
(2) *the period of any periodic point is power of two,*
(3) $UR(\phi) = Rec(\phi)$,
(4) $AP(\phi) = \{x \in I : \lim_{n\to\infty} \phi^{2^n}(x) = x\}$.

Note that the previous characterization of the dynamical simplicity is given in terms of the property "to have zero topological entropy". From a dynamics point of view when a discrete dynamical systems has zero topological entropy its dynamics is simple, i.e., is not chaotic, and therefore predictions on its future behaviour can be done in some sense, see [14]. From a dynamic point of view this type of result is the best possible, because we can check if its dynamic is simple or not by confirming the validity of one of the properties (1)–(4).

1.1. More than two firms

While dynamic properties of duopolies have been extensively studied, adjustment dynamics in Cournot processes with more than two firms has received much less attention as a consequence of the difficulties which appear when studying discrete dynamical systems with dimension higher than two. The direct generalization of the Cournot duopoly situation is the Cournot oligopoly, i.e., consider n firms which produce an identical good and in each step of the process any firm knows the among of product generated by the $n - 1$ rival firms in the previous step. Now, the systems which models the situation is given by

$$F(x_1, x_2, \dots, x_n) = (f_1(x_2, x_3, \dots, x_n), f_2(x_1, x_3, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_{n-1})) \tag{2}$$

where $f_i : [0, 1]^{n-1} \to [0, 1]$ is a continuous map. We note that the reaction function f_i depends on $n - 1$ variables of indices $j \in \{1, 2, \dots, n\}$, $j \neq i$.

For a system like (2) does not exist a characterization of dynamical simplicity in the way of Theorem 1 and is far from being obtained by the ignorance of the topological dynamics of n-dimensional discrete dynamical systems with $n > 2$ (e.g., note that for these type of systems are not characterized the possible ω-limit sets of the orbits).

Thus, if we want to have a tool for measuring the dynamical complexity, we need to simplify the system with the cost of loosing information by the firms on the production level of the rivals.
In [15] is introduced the following model called Cournot-like system:

Definition 2. A continuous map ϕ from $[0, 1]^n$ into itself is Cournot-like if it is of the form:

$$\phi(x_1, x_2, \dots, x_n) = \left(\phi_{\sigma(1)}(x_{\sigma(1)}), \dots, \phi_{\sigma(n-1)}(x_{\sigma(n-1)}), \phi_{\sigma(n)}(x_{\sigma(n)})\right),$$

where $\phi_i : [0, 1] \to [0, 1]$ is continuous, $i \in \{1, 2, \dots, n\}$ and σ is a cyclic permutation of the set $\{1, 2, \dots, n\}$.

In the economic situation models by these type of models the level of information is quite limited because any player firm only has information on the production level of one of the other firms in the previous step of the process. For these type of systems in [15] is proved that Theorem 1 works.

From our point of view Cournot-like models do not represent a real economic situation since it is very difficult to explain the fact that each player firm only can has information on other firm having a complete ignorance on the rest of firms behaviour. For that reason in the next section we introduce a new model where the information level is higher than in Cournot-like ones and where likely it is possible to obtain a characterization for the dynamical complexity.

2. Second extension: Local competition "à la Cournot"

The aim of this section is to introduced a new model: let $N = \{1, 2, \dots, n\}$ be the set of firms (i.e., rival firms which produce an identical good) and assume that are physically located around a circle or a line. We assume that the firms compete "à la Cournot" in a local way, i.e., each firm $i \in N$ competes with its closest neighboring in the right and left direction. Let $B_i^\alpha \subset N$ be the neighboring located to distance equal to α of the firm i in the right and left direction. If we