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Determination of Stackelberg–Nash equilibria using a sensitivity based approach

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ABSTRACT

A sensitivity based approach is presented to determine Nash solution(s) in multiobjective problems modeled as a non-cooperative game. The proposed approach provides an approximation to the rational reaction set (RRS) for each player. An intersection of these sets yields the Nash solution for the game. An alternate approach for generating the RRS based on design of experiments (DOE) combined with response surface methodology (RSM) is also explored. The two approaches for generating the RRS are compared on three example problems to find Nash and Stackelberg solutions. For the examples presented, it is seen that the proposed sensitivity based approach (i) requires less computational effort than a RSM-DOE approach, (ii) is less prone to numerical errors than the RSM-DOE approach, (iii) has the ability to find multiple Nash solutions when the Nash solution is not a singleton, (iv) is able to approximate nonlinear RRS, and (v) on one example problem, found a Nash solution better than the one reported in the literature.

1. Introduction

Multi objective optimization (MOO) problems requiring a simultaneous consideration of two or more conflicting objective functions frequently arise in design. Several approaches such as the weighted sum method, goal programming, multi-attribute utility theory, game theory, etc. have been used to solve multi-objective optimization problems. Game theory is one of the approaches that have been used to solve MOO problems where each player corresponds to an objective being optimized. The players control a subset of design variables and seek to optimize their individual payoff functions.

There are three types of games in the context of engineering design: cooperative game, non-cooperative (Nash) game, and an extensive game. In a cooperative game, the players have knowledge of the strategies chosen by other players and collaborate with each other to find a Pareto-optimal solution. If a cooperation or coalition among the players is not possible, the players make decision by making assumptions about unknown strategies selected by other players. In such situations, the rational reaction set [1] helps a decision maker (DM) choose a strategy based on unknown information from other DMs.

In a non-cooperative game, each player has a set of variables under his control and optimizes his objective function individually. The player does not care how his selection affects the payoff functions of other players. The players bargain with each other to obtain an equilibrium solution, if one exists. This solution is called Nash solution. Vincent [2] proposed the use

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of Nash solutions in solving design problems. Rao [3] extended the idea of the Nash solution from a problem with two players to a game with n players. Additional applications on the use of game theory in the context of mechanical design can be found in Badhrinath and Rao [4], Rao et al. [5], Lewis and Mistree [6], and Hernandez and Mistree [7].

Several approaches have been proposed over the years for the computation of Nash solutions in game-theoretic formulations. These include methods based on Nikaido–Isoda function [8], rational reaction set [6] and monotonicity analysis [5]. Recently, Shiau and Michalek [9] developed an optimization method which considers competitors' pricing reactions to new product design. They found that designs accounting for competitor reactions by imposing Nash and Stackelberg conditions as constraints are better than solutions than ignoring them altogether.

For some problems arising in mechanical design such as the pressure vessel problem considered in Rao et al. [5], closed form expressions for Nash equilibria can be obtained using the principles of montonicity analysis [10]. However, in general, numerical techniques are needed to approximate the RRS. A design of experiments based approach [11] coupled with response surface methodology [12] has been proposed by Lewis and Mistree [6], Marston [13], and Hernandez and Mistree [7]. This approach has been used by the authors to obtain Nash solutions for non-cooperative games as well as Stackelberg games.

This paper presents an alternate approach for obtaining Nash solutions that utilizes a sensitivity based formulation. The sensitivity of optimum solution to problem parameters has been explored by Sobieski et al. [14] and Hou [15]. This idea is adapted herein to construct the RRS for Nash and Stackelberg solutions. Results for three example problems with two or more objectives, and isolated as well as non-isolated Nash solutions are presented. From the example problems considered, it is seen that the proposed approach for constructing the RRS is computationally more efficient than RSM-DOE techniques because of its lower computational burden, its ability to find multiple Nash solutions, and on one example problem, yielding a Nash solution better than the one reported in the literature.

2. Rational reaction set and nash solution

Consider two players, 1 and 2, who select strategies x and y where $x \in X \subset R^{n_1}$ and $y \in Y \subset R^{n_2}$. Here X and Y are the set of all possible strategies each player can select. Let U denote the set of strategies which are feasible for the two players. The objective functions $f_1(x,y)$ and $f_2(x,y)$ represent the cost function for players 1 and 2, respectively.

The Nash game is a non-cooperative game where each player determines its set of optimum solutions based on the choices made by other player(s). This set of solutions for each player is called the rational reaction set (RRS). The RRS for players 1 and 2 are defined as follows:

$$f_1(x^N,y) = \min_{x \in X,} f_1(x,y) \rightarrow \quad x^N(y) \tag{1}$$

$$f_2(x,y^N) = \min_{y \in Y,} f_2(x,y) \rightarrow \quad y^N(x) \tag{2}$$

where x^N is the optimum solution of player 1 which varies depending on the strategy y chosen by player 2. The function $x^N(y)$ would be RRS for player 1. Similarly, $y^N(x)$ is the RRS of player 2. The intersection of these two sets, if it exists, is the Nash solution for the non-cooperative game. Therefore, if the parametric equations $x^N(y)$ and $y^N(x)$ are solved simultaneously, the resulting solution is the Nash solution (x^N, y^N).

Three points about Nash solutions are worth mentioning here. First, there might be bargaining games in which more than one Nash solution exists. This happens when the intersection of RRS of players 1 and 2 (Eqs. (1) and (2)) admits more than one solution. Second, given a solution (x^N, y^N), how can it be verified that it indeed is a Nash solution. Suppose player 1 knows the strategy of player 2, y^N, then player 1 asks the question: Can I improve my objective function by switching from x^N to another strategy? If every player answers no to this question, then (x^N, y^N) is a Nash solution. However, if any player answers yes, then the given solution is not a Nash solution. This simple test is based on the definition of the Nash equilibrium. Lastly, it is possible for one Nash solution to dominate another Nash solution.

3. Optimum solution sensitivity

Assume that the optimization problem of players 1 and 2 can be written as follows:

$$\begin{aligned}
&\text{Min}\ \underset{\text{by varying}\ x}{f_1(x,y)} \quad z \in R^n \\
&\text{subject to} \quad g_j^1(x,y) \leqslant 0 \quad j = 1,\ldots,n_g^1, \\
&\text{Min}\ \underset{\text{by varying}\ y}{f_2(x,y)} \quad z \in R^n
\end{aligned} \tag{3}$$