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Rough set theory applied to lattice theory

A.A. Estaji [a], M.R. Hooshmandasl [b], B. Davvaz [c,*]

[a] Department of Mathematics, Sabzevar Tarbiat Moallem University, Sabzevar, Iran
[b] Department of Computer Science, Yazd University, Yazd, Iran
[c] Department of Mathematics, Yazd University, Yazd, Iran

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ABSTRACT

In this paper, we intend to study a connection between rough sets and lattice theory. We introduce the concepts of upper and lower rough ideals (filters) in a lattice. Then, we offer some of their properties with regard to prime ideals (filters), the set of all fixed points, compact elements, and homomorphisms.

1. Introduction

The concept of a rough set was originally proposed by Pawlak [32] as a formal tool for modeling and processing incomplete information in information systems, also see [33,34]. Since then, the subject has been investigated in many studies. For example, see [10,12,13,22,26,28,36,37,40,41]. Rough set theory is an extension of set theory in which a subset of a universe is described by a pair of ordinary sets called lower and upper approximations. A key concept in Pawlak's rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. It soon invoked a natural question concerning a possible connection between rough sets and algebraic systems. Kuroki in [16] introduced the notion of a rough ideal in a semigroup. Kuroki and Wang [17] gave some properties of the lower and upper approximations with respect to the normal subgroups. Mordeson [29] used the covers of the universal set to define approximation operators on the power set of a given set. In [5,6], Davvaz dealt with a relationship between rough sets and ring theory and considered a ring as a universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring. In [14], Kazanci and Davvaz introduced the notions of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in a ring and gave some properties of such ideals, also see [8,9]. Rough modules have been investigated by Davvaz [6], also see [7,15,18,21,24,25,35]. In [44], the notions of rough prime ideals and rough fuzzy prime ideals in a semigroup were introduced. In [38], by considering the notion of an *MV*-algebra, Rasouli and Davvaz considered a relationship between rough sets and *MV*-algebra theory. They introduced the notion of rough ideal with respect to an ideal of an *MV*-algebra, which is an extended notion of ideal in an *MV*-algebra. In [39], rough approximations of Cayley graphs are studied, and rough edge Cayley graphs are introduced. Also, see [19,46–49].

* Corresponding author.
E-mail addresses: aa_estaji@yahoo.com (A.A. Estaji), hooshmandasl@yazduni.ac.ir (M.R. Hooshmandasl), davvaz@yazduni.ac.ir (B. Davvaz).

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We know from elementary arithmetic that for any two natural numbers a and b there is a largest number d which divides both a and b, namely the greatest common divisor $gcd(a, b)$ of a and b. Also, there is a smallest number m which is multiple of both a and b, namely the least common multiple $lcm(a, b)$. This is pictured in Fig. 1. Turning to another situation, given two statements a and b. There is a "weakest" statement implying both a and b, namely the statement "a and b", which we write as $a \wedge b$. Similarly, there is a "strongest" statement which is implied by a and b, namely "a or b", written as $a \vee b$. This pictured in Fig. 2. A third situation arises when we study sets A and B. Again, there is a largest set contained in A and B, the intersection $A \cap B$, and a smallest one containing both A and B, the union $A \cup B$. We obtain similar diagram in Fig. 3. It is typical for modern mathematics that seemingly different areas lead to very similar situations. Then, the idea is to construct the common features in these examples, to study these features, and to apply the resulting theory to many different areas.

In this paper, we discuss a general mathematical concept called a lattice, which includes all those examples and many others as special cases. The paper has addressed a connection between two research topics, namely, rough sets and lattice theory, both of which have applications across a wide variety of fields. The paper is in line with a series of algebraic studies reported in the literature. So far, a good number of researchers have dealt with a great host of aspects and issues in this field. In particular, the first papers on connections between rough sets and algebraic structures appeared in Information Sciences, for example [16,17], also see [4,5,21,44].

Lattice theory play an important role in many disciplines of computer science and engineering. For example, they have applications in distributed computing, concurrency theory, programming language semantics and data mining. They are also useful in other disciplines of mathematics such as combinatorics, number theory and group theory, see [1,3,20,23,27,30,31,42,43,45].

This paper is organized as follows: In Section 2, we review some basic notions and properties of lattice and rough set. In Section 3, the concept of upper rough ideal on a lattice is introduced and its basic properties are discussed. In Section 4, the concept of lower rough ideal on a lattice is introduced, and we show that $(\underline{Apr},\overline{Apr})$, where $\underline{Apr},\overline{Apr} : \mathcal{P}(L) \to \mathcal{P}(L)$ is a Galois connection and also if θ is a $\vee$-complete full equivalence relation on L and $\underline{Apr},\overline{Apr} : Id(L) \cup \{\emptyset\} \to Id(L) \cup \{\emptyset\}$, then $\left(\underline{Apr},\overline{Apr}\right)$ is a Galois connection.

2. Preliminaries of lattice and rough set

In this section, we make some general remarks on the concepts of a lattice and rough set theory. The following definitions and preliminaries are required in the sequel of our work and, hence, presented in brief.

A poset L is a *lattice* if and only if for every a and b in L both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist in L. Throughout this paper, L is a lattice with the least element 0 and the greatest element 1. For $X \subseteq L$ and $x \in L$ we write:

1. $\downarrow X = \{y \in L : y \leqslant x \text{ for some } x \in X\}$,
2. $\uparrow X = \{y \in L : y \geqslant x \text{ for some } x \in X\}$,
3. $\downarrow x = \downarrow\{x\}$,
4. $\uparrow x = \uparrow\{x\}$.

   We also say that:

5. X is a lower set if and only if $X = \downarrow X$,
6. X is an upper set if and only if $X = \uparrow X$.

A subset D of L is *directed* if it is non-empty and every finite subset of D has an upper bound in D (aside from non-emptiness, it is sufficient to assume that every pair of elements in L has an upper bound in L). Dually, we call a non-empty subset F of L *filtered* if every finite subset of F has a lower bound in F. We also say F is a *filter* if and only if it is a filtered upper set.

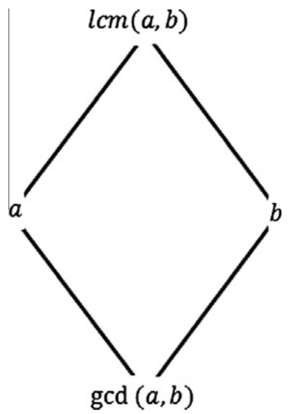

Fig. 1. a, b are two natural numbers, $gcd(a, b)$ is the greatest common divisor of a, b, and $lcm(a, b)$ is the least common multiple of a, b.