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Rough set theory based on two universal sets and its applications

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ABSTRACT

For two universal sets U and V, we define the concept of solitary set for any binary relation from U to V. Through the solitary sets, we study the further properties that are interesting and valuable in the theory of rough sets. As an application of crisp rough set models in two universal sets, we find solutions of the simultaneous Boolean equations by means of rough set methods. We also study the connection between rough set theory and Dempster–Shafer theory of evidence. In particular, we extend some results to arbitrary binary relations on two universal sets, not just serial binary relations. We consider the similar problems in fuzzy environment and give an example of application of fuzzy rough sets in multiple criteria decision making in the case of clothes.

1. Introduction

Rough set theory was developed by Pawlak as a formal tool for representing and processing information in database. In Pawlak rough set theory [19,20], the lower and upper approximation operators are based on equivalence relation. However, the requirement of an equivalence relation in Pawlak rough set models seem to be a very restrictive condition that may limit the applications of the rough set models. Thus one of the main directions of research in rough set theory is naturally the generalization of the Pawlak rough set approximations. For instance, the notations of approximations are extended to general binary relations [1,11,12,22,25,32–34,37], neighborhood systems [6], coverings [36], completely distributive lattices [2], fuzzy lattices [16] and Boolean algebras [14,23].

On the other hand, the generalization of rough sets in fuzzy environment is another topic receiving much attention in recent years [4–9,21,24]. Based on equivalence relation, the concept of fuzzy rough set was first proposed by Dubois and Prade [4] in the Pawlak approximation space. Yao [33] gave a unified model for both rough fuzzy sets and fuzzy rough sets based on the analysis of level sets of fuzzy sets. Pei [3,24] considered the approximation problems of fuzzy sets in fuzzy information systems results in theory of fuzzy rough sets. Li and Zhang [5] analyzed crisp binary relations and rough fuzzy approximations.

Rough set models on two universal sets can be interpreted by both generalized approximation spaces and the notions of interval structures [29]. Much research [5,24,30] has been done for these models. In this paper, we attempt to conduct a further study along this line.

This paper is organized as follows: In Section 3, for two universal sets U and V, we define the concept of solitary set for any binary relation from U to V. Through the solitary sets, we list the further properties that are interesting and valuable in the theory of rough sets. In Section 4, as an application of crisp rough set models in two universal sets, we have found an algorithm for the simultaneous Boolean equations by means of rough set methods. In Section 5, we study the relationship between rough set theory and Dempster–Shafer theory of evidence. In particular, we extend some results to arbitrary binary relations on two universal sets, not just serial binary relations. In Section 6, we study the basic properties of fuzzy rough sets in two universal sets. In Section 7, we give a case for the multiple criteria decision making on choices of clothes designs. Section 8 concludes the paper.

2. Preliminaries

Wong et al. [29] generalized the rough set models using two distinct but related universal sets. Let U and V denote two universal sets of interest and R a binary relation which is a subset of the Cartesian product $U \times V$. We called the triplet (U, V, R) an approximation space. $P(U), P(V)$ denote the power sets of U, V, respectively.

With respect to R, we define right neighborhood $r(x)$ of an element x in U, the R-relative set of x in U, to be the set of y in V with the property that x is R-related to y. Thus, in symbols, $r(x) = \{y \in V | xRy\}$. Similarly, the left neighborhood $l(y)$ of an element y in V is $l(y) = \{x \in U | xRy\}$.

By the notation $r(x)$, elements in U may be viewed as equivalent if they have the same R-relative set. Thus, an equivalence relation between the elements can be formally defined. Let E_U denote the equivalence relation on U.

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Definition 2.1 [15]. Two elements x and x' in the universal set U are equivalent if $r(x) = r(x')$, that is, xE_Ux' if and only if $r(x) = r(x')$.

The equivalence relation E_U partitions the set U into disjoint subsets. We use $[x]_E$ to denote an equivalence class in U containing an element $x \in U$. Similarly, the equivalence relation E_V on V can be defined by: yE_Vy' if and only if $l(y) = l(y')$.

Suppose that U, V and W are universal sets, R is a relation from U to V, and S is a relation from V to W. We can then define a relation, the composition of R and S, written as $S \circ R$. The relation $S \circ R$ is a relation from U to W and is defined as follows. If a is in U and c is in W, then $a(S \circ R)c$ if and only if for some b in V, we have aRb and bSc. Binary relations R, E_U, E_V have the following properties.

Proposition 2.1 [15]. *Let (U, V, R) be the approximation space, E_U, E_V as above, then $E_V \circ R = R = R \circ E_U$.*

With the approximation space (U, V, R), we define the lower and upper approximation operators $\underline{R}, \overline{R} : P(V) \to P(U)$ by [34]

$$\underline{R}Y = \{x \in U | r(x) \subseteq Y\}, \quad \text{and} \quad \overline{R}Y = \{x \in U | r(x) \cap Y \neq \emptyset\},$$

respectively. The pair $RY = (\underline{R}Y, \overline{R}Y)$ is referred to as the rough set of $Y \in P(V)$.

Note that the lower and upper approximation operators $\underline{R}, \overline{R}$ can also be represented by equivalence class $[x]_E$ as follows:

$$\underline{R}Y = \cup_{r(x) \subseteq Y}[x]_E \quad \text{and} \quad \overline{R}Y = \cup_{r(x) \cap Y \neq \emptyset}[x]_E.$$

Example 2.1. Let $U = \{x_1, x_2, x_3, x_4\}, V = \{y_1, y_2, y_3, y_4, y_5\}$, relation R is given by its corresponding matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then $U/E_U = \{\{x_1\}, \{x_2, x_4\}, \{x_3\}\}, V/E_V = \{\{y_1, y_2\}, \{y_3\}, \{y_4\}, \{y_5\}\}$. Let $Y = \{y_1, y_2, y_5\}$, then $\overline{R}Y = \{x_1, x_2, x_4\}, \underline{R}Y = \{x_1\}$.

3. Algebraic properties of rough sets

This section studies the algebraic structure of the lower and upper approximation operators of rough sets in two universal sets. The results provide a better understanding of rough sets.

A very useful concept for sets is the characteristic function. If X is a subset of a universal set U, the characteristic function of X, still denoted by X, is defined for each $x \in U$ as follows:

$$X(x) = \begin{cases} 1, & x \in X \\ 0, & x \notin X \end{cases}$$

For the arbitrary binary relation R from U to V, $R(x, y)$ is defined by

$$R(x, y) = \begin{cases} 1, & (x, y) \in R \\ 0, & (x, y) \notin R \end{cases}$$

Rough set lower and upper approximations can be restated by characteristic functions as follows:

Proposition 3.1. *Let U, V be two universal sets, R a binary relation from U to V. The lower and upper approximations can be written as: $\forall Y \in P(V), \forall x \in U$*

(1) $(\overline{R}Y)(x) = \vee_{y \in r(x)} Y(y)$;
(2) $(\underline{R}Y)(x) = \wedge_{y \in r(x)} Y(y)$.Where $\wedge$ denotes minimum and $\vee$ maximum.

Proof. (1) We only need to prove that $(\overline{R}Y)(x) = 1$ if and only if $\vee_{y \in r(x)} Y(y) = 1$. Indeed, if $(\overline{R}Y)(x) = 1$, then $x \in \overline{R}Y$, hence there exist some $z \in V$ such that $z \in r(x) \cap Y$; that is, $z \in r(x), Y(z) = 1$, therefore $\vee_{y \in r(x)} Y(y) = 1$, and vice versa. A similar argument works to prove part (2). □

Furthermore, if universal sets U and V are finite sets, then a binary relation from U to V can be represented by a Boolean matrix and a subset of V can be represented by a column Boolean vector. Note that Part (1) of Proposition 3.1 can be restated as the following equivalent form:

$$(\overline{R}Y)(x) = \vee_{y \in V}(R(x, y) \wedge Y(y)).$$

We have the following proposition.

Proposition 3.2. *Let $U = \{x_1, x_2, \ldots, x_m\}$ and $V = \{y_1, y_2, \ldots, y_n\}$ be two finite universal sets, R a binary relation form U to V, M_R the $m \times n$ matrix representing R, and Y a subset of V. Then*

$$\overline{R}Y = M_R \circ Y,$$

where $M_R \circ Y$ is the Boolean product of $m \times n$ matrix M_R and column Boolean vector Y.

Definition 3.1 [17]. Let U, V be two universal sets and R be a binary relation from U to V, if $x \in U$ and $r(x) = \emptyset$, we call x a solitary element with respect to R. The set of all solitary elements with respect to R is called the solitary set and denoted as S, i.e.,

$$S = \{x | x \in U, r(x) = \emptyset\}.$$

Recall that a binary relation on U is called a serial relation if every element $x \in U$, has at least one element $y \in U$ such that xRy. In other words, $S = \emptyset$.

Through the solitary set, we list the algebraic properties that are interesting and valuable in the theory of rough sets as follows:

Proposition 3.3. *Let R be an arbitrary binary relation from U to V and S be the solitary set respect to R. Then the lower and upper approximation operators satisfy the following properties: for subsets X, Y in V,*

(1) $\overline{R}Y = \cup_{y \in Y} l(y)$;
(2) $\underline{R}\emptyset = S$, $\overline{R}\emptyset = \emptyset$, $\underline{R}V = U$ and $\overline{R}V = S^C$, where S^C denotes the complement of S in U;
(3) $S \subseteq \underline{R}X$ and $\overline{R}X \subseteq S^C$;
(4) $\underline{R}X - S \subseteq \overline{R}X$;
(5) $\underline{R}X = U$ if and only if $\cup_{x \in U} r(x) \subseteq X$, $\overline{R}X = \emptyset$ if and only if $X \subseteq (\cup_{x \in U} r(x))^C$;
(6) If $S \neq \emptyset$, then $\underline{R}X \neq \overline{R}X$ for all $X \in P(V)$;
(7) For any given index set $I, X_i \in P(V)$, $\underline{R}(\cap_{i \in I} X_i) = \cap_{i \in I} \underline{R}X_i$ and $\overline{R}(\cup_{i \in I} X_i) = \cup_{i \in I} \overline{R}X_i$;
(8) If $X \subseteq Y$, then $\underline{R}X \subseteq \underline{R}Y$ and $\overline{R}X \subseteq \overline{R}Y$;
(9) $\underline{R}X \cup \underline{R}Y \subseteq \underline{R}(X \cup Y)$, and $\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y$;
(10) $(\underline{R}X)^C = \overline{R}X^C$, and $(\overline{R}X)^C = \underline{R}X^C$;
(11) There exists some $X \in P(U)$ such that $\overline{R}X = \underline{R}X$ if and only if R is serial.
(12) If S is another binary relation from U to V and $\overline{R}X = \overline{S}X$ for all $X \in P(V)$, then $R = S$.
(13) If S is another binary relation from U to V and $\underline{R}X = \underline{S}X$ for all $X \in P(V)$, then $R = S$.

Axiomatic approach [10,13] is significant in rough set theory. The axiomatic approach aims to investigate the mathematical characters of rough sets, which may help to develop methods for applications. Now we present an axiomatic system for rough sets on two universal sets.