Insurance: Mathematics and Economics 44 (2009) 426–433

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal risk sharing with different reference probabilities

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ARTICLE INFO

Article history:
Received July 2008
Received in revised form
December 2008
Accepted 25 December 2008

IME Subject and Insurance Branch Category:
IM51
IE12

JEL classification:
D81
G22

MSC:
91B06
91B16
91B30

Keywords:
Optimal risk sharing
Law-invariance
Convolution

ABSTRACT

We investigate the problem of optimal risk sharing between agents endowed with cash-invariant choice functions which are law-invariant with respect to different reference probability measures. We motivate a discrete setting both from an operational and a theoretical point of view, and give sufficient conditions for the existence of Pareto optimal allocations in this framework. Our results are illustrated by several examples.

1. Introduction

The optimal exchange of risk between two parties is one of the major issues in mathematical economics and finance, and many authors have studied this problem, since the early works of Arrow (1963), Borch (1962) and Du Mouchel (1968), where the risk sharing is analyzed in the insurance and reinsurance context. The introduction of concepts like coherent and convex risk measures, by Artzner et al. (1999) and Föllmer and Schied (2002), recently paved the way to a further analysis of the problem of optimal sharing and allocation of risk. Several authors have considered the exchange of risk between agents endowed with these kinds of choice functions (see, e.g., Deprez and Gerber (1985), Chateauneuf et al. (2000), Barrieu and El Karoui (2005), Jouini et al. (2008), Filipović and Kupper (2007, 2008) and Burgert and Rüschendorf (2008, 2006)) or in a slightly more general setting (see, e.g., Acciaio (2007) and Filipović and Svindland (2008)).

A natural property to require on those choice functions is indifference with respect to financial positions having the same distribution under some reference probability measure. This is the so-called law-invariance property, studied, e.g., by Kusuoka (2001), Frittelli and Rosazza Gianin (2005) and Jouini et al. (2006). When all choice functions are assumed to be cash-invariant (or translation invariant) and law-invariant with respect to the same reference probability measure, the existence of optimal allocations has already been proved, see Jouini et al. (2008), Filipović and Svindland (2008) and Acciaio (2007). In this paper we study the risk sharing problem in the situation of two economic agents with different views of the world, that is, with different (subjective) reference probability measures. We consider them equipped with cash-invariant choice functions which are law-invariant in their respective worlds, i.e., with respect to their different reference probabilities. Manifold causes may motivate such a framework. In case of financial corporations, for instance, these different world views might stem from different internal models, from having access to different information, or from being subject to guidelines

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doi:10.1016/j.insmatheco.2008.12.002

of different regulating agencies. We can consider, for example, the case of financial firms subject to stress tests, possibly managed by different supervising agencies. These tests are made to gauge the potential vulnerability of the firms to a given set of particular market events or stress scenarios, like a stock market crash or other market shocks. In this situation firms are interested in maintaining a 'good' position in case one of those shock scenarios were to occur, in order to 'pass' the test. This might amongst other things be achieved by exchanging risk in this set of events. We like to point out that our setting is in particular valid in a situation of both competition and cooperation. It is often observed that competing agents in a market do cooperate on the level of exchanging certain risks if they all believe to benefit from it. However, since in the end the agents are competitors, it is very unlikely that they will share full information with each other, that is agree on one reference probability, and thereby abandon advantages over their competitors which might result from access to some particular information. E.g. in insurance companies or banks the internal model is a crucial competitive advantage which is kept secret from other agents in the market. Within one firm, i.e. within one internal model, law-invariance of the choice function used is very likely. Hence, we end up in a situation as described above where the incomplete and different information is represented by the different reference probabilities of the agents which are a priori unrelated. Nevertheless, these competing agents may all benefit from trading/exchanging some of their risks on some prominent events which are common knowledge, like increase or decrease of interests, natural hazards, etc., but which are weighted differently according to different information.

In any case, any exchange of risk requires some kind of cooperation between the involved agents, who will have to agree on some set of (prominent) scenarios on which they want some level of protection. As described above it is very likely that the agents weight these scenarios in a different way depending on their respective information. Moreover, it turns out that, in practice, such a set of scenarios is usually finite, thus reducing the optimal risk allocation problem to a finite dimensional risk exchange on these base scenarios. In this framework, and under some mild additional conditions, we show that there always exist Pareto optimal allocations for any aggregate risk. Our results are illustrated by several examples. In particular, we also give examples which show that Pareto optimal allocations do not exist in general if we drop some of our assumptions (Examples 4.3 and 4.4).

The remainder of the paper is organized as follows. In Section 2 we formalize the optimal risk sharing problem, and we state our main result on the existence of optimal allocations (Theorem 2.3). This result is then proved in Section 3. Our examples are collected in Section 4. We assume the reader to be familiar with basic convex duality as outlined in Rockafellar (1997) or Ekeland and Témam (1999). However, in the Appendix we give a short review on some basic concepts and notation from convex analysis which are frequently used throughout the paper. Some known results are also postponed to the Appendix.

2. Optimal risk sharing problem

2.1. Framework

We consider a measurable space $(\Omega, \mathcal{F})$ and two probability measures $\mathbb{P}_1, \mathbb{P}_2$ on $(\Omega, \mathcal{F})$ such that $(\Omega, \mathcal{F}, \mathbb{P}_i)$, $i = 1, 2$, are non-atomic standard probability spaces. The measures $\mathbb{P}_1, \mathbb{P}_2$ describe the view of two agents, say 1 and 2, on the world $(\Omega, \mathcal{F})$. The preferences of the i-th agent on $L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ are represented by a choice function $U_i : L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i) \to \mathbb{R}$, that throughout the paper is assumed to satisfy the following conditions:

(C1) concavity: $U_i(\alpha X + (1-\alpha)Y) \geq \alpha U_i(X) + (1-\alpha)U_i(Y)$ for all $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ and $\alpha \in (0, 1)$;

(C2) cash-invariance: $U_i(X + c) = U_i(X) + c$ for all $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ and $c \in \mathbb{R}$;

(C3) normalization: $U_i(0) = 0$;

(C4) $\mathbb{P}_i$-law-invariance: $U_i(X) = U_i(Y)$ whenever $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ are identically distributed under $\mathbb{P}_i$;

(C5) upper semi-continuity (u.s.c.): for any sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ converging to some $X \in L^\infty$, we have $U_i(X) \geq \limsup_n U_i(X_n)$.

If U_i in addition is monotone, i.e. $U_i(X) \geq U_i(Y)$ whenever $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ satisfy $X \geq Y \mathbb{P}_i$-a.s., then U_i is a $\mathbb{P}_i$-law-invariant monetary utility function, i.e. $-U_i$ is a $\mathbb{P}_i$-law-invariant convex risk measure in the sense of Föllmer and Schied (2004). Note that (C5) is equivalent to the continuity of U_i because U_i is finitely-valued (see e.g. Ekeland and Témam (1999) Corollary 2.5), and that any proper function on $L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ which is monotone and satisfies (C2) is automatically finitely-valued and 1-Lipschitz-continuous (see Föllmer and Schied (2004)). It is proved in Jouini et al. (2006) that the $\mathbb{P}_i$-law-invariance ensures the following dual representation for U_i (the so-called Fatou property):

$$U_i(X) = \inf_{Z\in L^1(\Omega,\mathcal{F},\mathbb{P}_i)} \{V_i(Z) + E[ZX]\}, \quad X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i), \tag{2.1}$$

where V_i is the dual U_i^* of U_i (see (B.1)). Note how this representation of U_i can be seen as a worst-case evaluation, based on the set of test measures $\mathcal{Q}_i = \{\mathbb{Q}\ \sigma\text{-additive measure} : \mathbb{Q} \ll \mathbb{P}_i, V_i(\frac{d\mathbb{Q}}{d\mathbb{P}_i}) < \infty\}$, where to each test measure $\mathbb{Q} \in \mathcal{Q}_i$ is associated a penalty $V_i(\frac{d\mathbb{Q}}{d\mathbb{P}_i})$ which expresses the confidence of agent i on $\mathbb{Q}$ (the higher the penalty, the lower the confidence on that measure).

Here we fix the space $L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ of $\mathbb{P}_i$-essentially-bounded random variables as the set of possible financial positions considered by agent i. However, we can also think of the choice functions as defined on $L^{p_i}(\Omega, \mathcal{F}, \mathbb{P}_i)$, for any p_i in $[1, \infty]$ and possibly $p_1 \neq p_2$. Note that we do not require the $\mathbb{P}_i$'s to fulfill any absolute continuity- or even equivalence-relation. A priori the world views $\mathbb{P}_1$ and $\mathbb{P}_2$ are unrelated.

2.2. Formulation of the problem

The problem we address in this paper is the optimal sharing and allocation of risk between two agents who have different views of the world, in the sense described in Section 2.1. As motivated in the Introduction, the discrete setting turns out to be a proper framework to formulate this problem. Roughly speaking, no matter how different is the world view of the two agents, we assume they agree on a finite set of possible scenarios. Therefore, any information they have about the preferences of the other, and any risk they consider or exchange, is relative to this set. To put this into mathematical terms, we let $A = \{A_1, \ldots, A_n\} \subset \mathcal{F}$ be a finite partition of Ω and $\mathcal{F}_A := \sigma(\{A_1, \ldots, A_n\})$ the σ-algebra it generates. The A_j's are the base events on which agents agree to exchange risk, and we assume that

$$\mathbb{P}_i(A_j) > 0 \quad \text{for all } j = 1, \ldots, n \text{ and } i = 1, 2. \tag{2.2}$$

This latter condition does not only seem natural, but it is in fact necessary for the existence of optimal allocations. Indeed, assume $0 = \mathbb{P}_1(A_j) < \mathbb{P}_2(A_j)$ (or vice versa, mutatis mutandis) for some $j \in \{1, \ldots, n\}$, then agent 2 could increase her wealth on A_j as much as she likes, and agent 1 would take all the risk on A_j. Hence, in this situation there cannot be any optimum. Moreover, let $\mathbb{Q}^+$ be the set of positive rationals, we assume that

$$\mathbb{P}_1(A_j) \in \mathbb{Q}^+ \quad \text{for all } j = 1, \ldots, n, \tag{2.3}$$

which is no restriction in the interesting cases. A finite partition $A = \{A_j\}_{j=1}^n$ of Ω such that $\{A_j\}_{j=1}^n \subset \mathcal{F}$ and (2.2), (2.3) hold will be called admissible. Now let $A = \{A_j\}_{j=1}^n$ be an admissible partition of Ω. Then the space of admissible financial positions which the